Compound interest

Conventional time deposit with pay-off period 3 years

Task 1: A businessman deposited onto conventional time deposit the amount of CZK 300,000.00. What will be the amount of capital in 3 years if interest rate amounts to 2% p.a. in

a) yearly interest period,
b) half-yearly interest period,
c) quarterly interest period?

The interest is credited to the deposit and then interest is paid on the deposit

Solution:
For the amount K_n saved after n years at m interest periods for one year, formula

$K_n = K_0 \cdot \left(1 + \frac{i}{m}\right)^{m \cdot n}$ holds true, where where K0 is initial amount and i is annual interest rate (expressed by decimal number). In case that interest accrued is entered once a year (i.e. m = 1), we get a more simple formula in form is initial amount and i is annual interest rate (expressed by decimal number). In case that interest accrued is entered once a year (i.e. $m = 1$), we get a more simple formula in form $K_n = K_0 \cdot (1 + i)^n$.

Derivation of formula $K_n = K_0 \cdot \left(1 + \frac{i}{m}\right)^{m \cdot n}$

Firstly we derive the relation for the case of half-yearly entering of interest accrued (m = 2) in 3 years (n = 3):

```
> restart;
> K11:=factor(K0+K0*i/2);    K12:=factor(K11+K11*i/2);
```

$$K11 := \frac{1}{2}\, K0\, (2 + i)$$

$$K12 := \frac{1}{4}\, K0\, (2 + i)^2 \quad \textbf{(1.1)}$$

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> K21:=factor(K12+K12*i/2);   K22:=factor(K21+K21*i/2);
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$$K21 := \frac{1}{8}\, K0\, (2 + i)^3$$

$$K22 := \frac{1}{16}\, K0\, (2 + i)^4 \quad \textbf{(1.2)}$$

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> K31:=factor(K22+K22*i/2);   K32:=factor(K31+K31*i/2);
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$$K31 := \frac{1}{32} K0 (2+i)^5$$

$$K32 := \frac{1}{64} K0 (2+i)^6 \qquad \textbf{(1.3)}$$

We can see that exponent equals 6 = 2.3 and denominator is 64 = 26, which corresponds with the formula for K_n. It is evident that generalization will result in formula $K_n = K_0 \cdot \left(1 + \frac{i}{m}\right)^{m \cdot n}$

The formula for K_n is defined in Maple as the function with variables K0, i, n, m:

```
restart;
```

$K := (K0, i, n, m) \rightarrow K0 \cdot \left(1 + \frac{i}{m}\right)^{m \cdot n};$

$$(K0, i, n, m) \rightarrow K0 \left(1 + \frac{i}{m}\right)^{m n} \qquad \textbf{(1)}$$

as to a) annual interest period

The amount of saved capital is $K(300000, 0.02, 3, 1)$ = 318362,40 CZK

as to b) half-yearly interest period

The amount of saved capital is $K(300000, 0.02, 3, 2)$ = 318456,04 CZK

as to c) quarterly interest period

The amount of saved capital is $K(300000, 0.02, 3, 4)$ = 318503,34 CZK

Notes

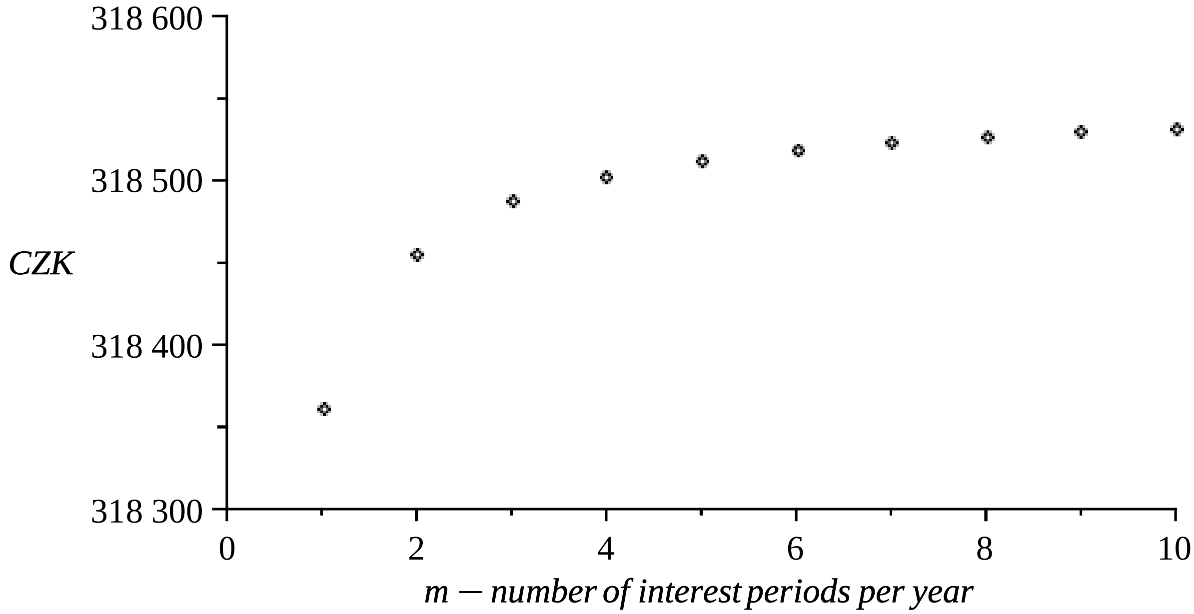
1. From task handling results we can see that the more often the interest accrued is entered, the higher evaluation of the amount deposited. The highest evaluation corresponds with the case when $m \rightarrow \infty$.
In case of our task, saved amount would achieve $limit(K(300000, 0.02, 3, m), m = infinity)$ = 318550,96 CZK.
In general it holds true that

$Limit(K(K0, i, n, m), m = infinity) = limit(K(K0, i, n, m), m = infinity);$

$$\lim_{m \rightarrow \infty} \left(K0 \left(1 + \frac{i}{m}\right)^{m n}\right) = e^{i n} K0 \qquad \textbf{(2)}$$

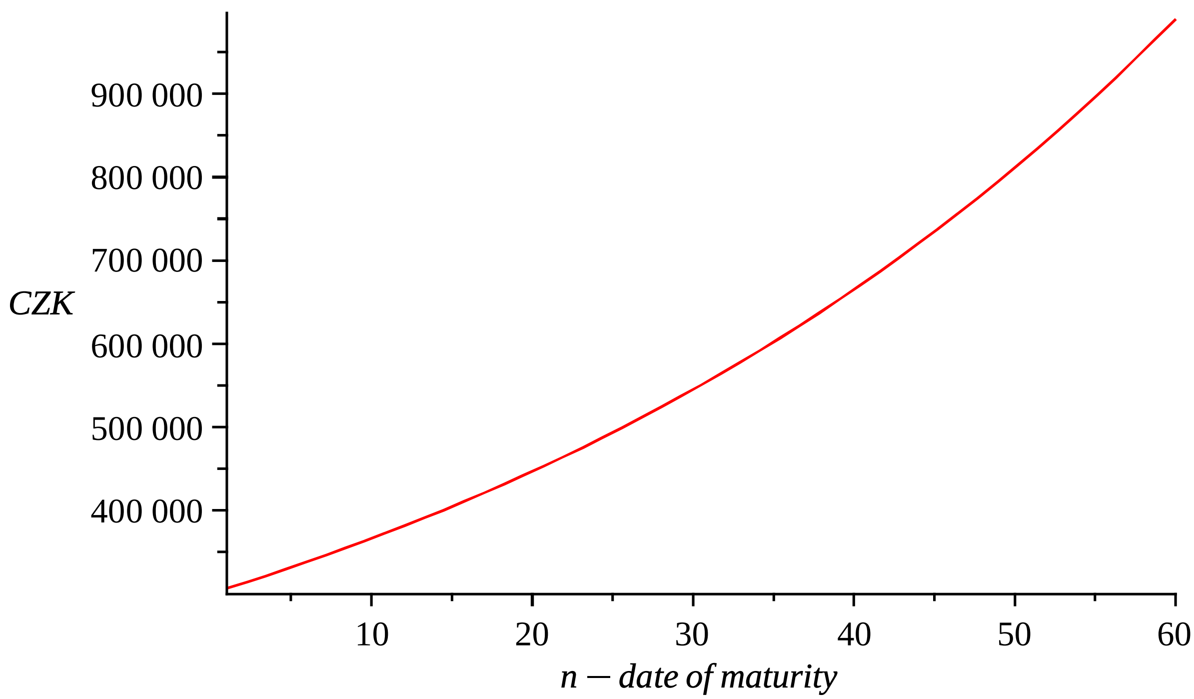
2. By comparison of the results under a), b) and c) we can see that a change in value m from 1 to 2 results in a greater change in saved capital than a change in value m from 2 to 4. This is a generally tru property - the biggest financial jump will occur in a change of interest period from one yeat to half a year. This is illustrated by the graphb as follows:

```
plots[pointplot]([seq([m, K(300000, 0.02, 3, m)], m = 1..10)], view = [0..10, 318300..318600],
    labels = [m - number of interest periods per year, CZK]);
```



3. It is evident that the longer the payment period *n*, the higher the growth of saved amount.This is illustrated by graphical representation of dependence K_n on n at fixed $m = 2$:

plot(*K*(300000, 0.02, *n*, 2), *n* = 1 ..60, *labels* = [*n* - *date of maturity*, *CZK*]);



From the above graph we can see that an increasing payment period results in exponential growth of the amount of capital.

Tasks

Practical problems concerning time deposits are settled by means of equation $K_n = K_0 \cdot \left(1 + \frac{i}{m}\right)^{m \cdot n}$, possibly $K_n = K_0 \cdot (1 + i)^n$, where one parameter is always unknown and the remaining parameters are known.

Task 1: What amount shall we invest in a guaranteed bond fund in order to assure for now a 14-year old child at his/her 19 years the amount CZK 300,000.00 for university studies. Let us suppose payment period 5 years and that for this period the rate of return is 5% p.a. (We consider yearly entering of interest accrued).

Solution:

Let us suppose annual interest period, i.e. $m = 1$. From equation $K_n = K_0 \cdot (1 + i)^n$ we express K_0 and calculate its value:

restart;

$K0 := unapply\left(solve\left(Kn = K0 \cdot (1 + i)^n, K0\right), Kn, i, n\right);$

$$(Kn, i, n) \rightarrow \frac{Kn}{(1 + i)^n} \quad \textbf{(3)}$$

Answer:

We shall invest the amount $K0(300000, 0.05, 5) =$ 235057,85 CZK.

Task 2: What shall be the rate of return in order to assure for our 14-year old child at his/her age of 19 years the amount CZK 300,000.00, if presently we have available amount CZK 250,000.00?

Solution:

Let us suppose annual interest period, i.e. $m = 1$. From equation $K_n = K_0 \cdot (1 + i)^n$ we express i and calculate its value:

restart;

$i := unapply\left(simplify\left(solve\left(Kn = K0 \cdot (1 + i)^n, i\right)\right), Kn, K0, n\right);$

$$(Kn, K0, n) \rightarrow \left(\frac{Kn}{K0}\right)^{\frac{1}{n}} - 1 \quad \textbf{(4)}$$

Answer:

The rate of return shall be

$evalf(i(300000, 250000, 5), 4) \cdot 100 =$ 3.700 %.

Task 3: We have available amount CZK 200,000.00 and the possibility of its investing with the rate of return 4%. Let us suppose that the rate of return will not change in the future very much. We need that our newly-born child at 19 years will acquire CZK 300,000.00 for his/her studies. In what age of the child should we invest respective capital at the above rate of return?

Solution:

Let us suppose annual interest period, i.e $m = 1$. From equation $K_n = K_0 \cdot (1+i)^n$ we express n and calculate its value:

restart;
$n := unapply\left(simplify\left(solve\left(Kn = K0 \cdot (1+i)^n, n\right)\right), Kn, K0, i\right);$

$$(Kn, K0, i) \rightarrow \frac{\ln\left(\frac{Kn}{K0}\right)}{\ln(1+i)} \quad \textbf{(5)}$$

Answer:
If we invest the amount CZK 200,000.00 under the above conditions, we will get required amount CZK 300,000.00 for

$$evalf(n(300000, 200000, 0.04)) = 10.33803507 \text{ years.}$$

From this it is evident that at respective rate of return we shall invest capital at the child's age of 19 years -

$$evalf(n(300000, 200000, 0.04)) = 8.66196493 \text{ years.}$$