A GROUP OVER THE SET OF PASCAL POINTS ON THE SIDES OF A CONVEX QUADRILATERAL

DOV FRAIVERT AND DAVID FRAIVERT

Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. This theory defines the concepts of "pairs of Pascal points on the sides of a convex quadrilateral" and "a circle that forms Pascal points".

In the present paper, we shall find all the existing pairs of Pascal points and show that it is possible to define a group structure over the pairs of Pascal points on the sides of a convex quadrilateral.

Introduction: General concepts and theorems of the theory of a convex quadrilateral and a circle that forms Pascal points

First, we shall briefly survey the definitions of some essential concepts of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides. We shall also present the fundamental theorem and several selected properties of the theory (see [1], [2] , [3], [4]).

The theory considers the situation in which $ABCD$ is a convex quadrilateral for which there exists a circle ω that satisfies the following two requirements:

(i) Circle ω passes through point E, the point of intersection of the diagonals, and through point F, the point of intersection of the extensions of sides BC and AD.

(ii) Circle ω intersects sides BC and AD at interior points (points M and N, respectively, in Figure 1a.

The Fundamental Theorem of the theory holds in this case.

The fundamental theorem *Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the extensions of these sides, and that passes through the point of intersection of the diagonals.*
In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the extension of a diagonal.
Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.
(In Figure 1b, straight lines h and g intersect at point P on side AB, and straight lines i and j intersect at point Q on side CD).

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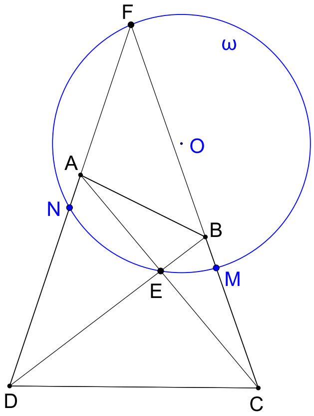

Figure 1

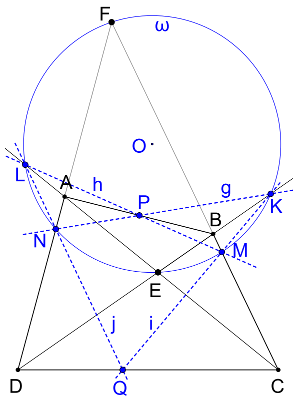

Figure 2

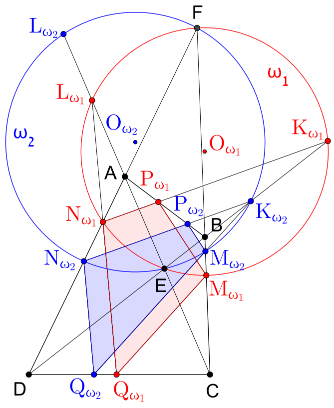

Figure 3

In the proof of this theorem, we prove that points A, P, and B are collinear using Pascal's theorem for the crossed hexagon $EKNFML$, and we also prove that points C, Q, and D are collinear using Pascal's theorem for the crossed hexagon $EKMFNL$ (see [1]).

Definitions.

Because the proof of the properties of the points of intersection P and Q is based on Pascal's Theorem:

(i) Points P and Q are termed ***"Pascal points" on sides AB and CD of the quadrilateral***.

(ii) The circle that passes through points of intersection E and F and through two opposite sides is termed ***"a circle that forms Pascal points on the sides of the quadrilateral"***.

We shall present several selected properties of the Pascal points on the sides of a convex quadrilateral. We shall use some of these properties below.

Given is: Let $ABCD$ be a convex quadrilateral in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD; let ω_i be some circle (whose center is O_{ω_i}) that passes through points E and F and through interior points M_{ω_i} and N_{ω_i} of sides BC and AD, respectively; and let P_{ω_i} and Q_{ω_i} be the Pascal points that are formed by ω_i on sides AB and CD, respectively.

Property 1: For every three circles, ω_1, ω_2, and ω_3, there holds (see [1, Theorems 2 and 5]): $\dfrac{P_{\omega_1}P_{\omega_2}}{P_{\omega_2}P_{\omega_3}} = \dfrac{Q_{\omega_1}Q_{\omega_2}}{Q_{\omega_2}Q_{\omega_3}} = \dfrac{O_{\omega_1}O_{\omega_2}}{O_{\omega_2}O_{\omega_3}}$.

Property 2: For every two circles, ω_1 and ω_2, there holds (see the proof of Theorem 1 in [1] and Theorem A in [2]): *The corresponding sides of quadrilaterals $P_{\omega_1}M_{\omega_1}Q_{\omega_1}N_{\omega_1}$ and $P_{\omega_2}M_{\omega_2}Q_{\omega_2}N_{\omega_2}$ are parallel to each other.* (see Figure 1c).

Property 3: *Among the circles ω_i there is a single circle, ω_a, for which Pascal points P_{ω_a} and Q_{ω_a} are collinear with center O_{ω_a}. For this circle, it holds that quadrilateral $P_{\omega_a}M_{\omega_a}Q_{\omega_a}N_{\omega_a}$ is a kite* (see [2]).

In the case that quadrilateral $ABCD$ is cyclic, the following additional properties hold:

Property 4: For every two circles, ω_1 and ω_2, there holds:
Quadrilaterals $P_{\omega_1}M_{\omega_1}Q_{\omega_1}N_{\omega_1}$ *and* $P_{\omega_2}M_{\omega_2}Q_{\omega_2}N_{\omega_2}$ *have equal perimeters* (see [2, Theorem 4]).

Property 5: For circle ω_{EF}, whose diameter is segment EF, there holds (see [2, Theorems 3 and 5]):

(i) *Points* $P_{\omega_{EF}}$ *and* $Q_{\omega_{EF}}$ *are the midpoints of sides* AB *and* CD*, respectively.*

(ii) *Quadrilateral* $P_{\omega_{EF}}M_{\omega_{EF}}Q_{\omega_{EF}}N_{\omega_{EF}}$ *is a kite* (see Figure 2a).

(iii) *Among all the quadrilaterals* $P_{\omega_i}M_{\omega_i}Q_{\omega_i}N_{\omega_i}$*, the one with the maximal area is* $P_{\omega_{EF}}M_{\omega_{EF}}Q_{\omega_{EF}}N_{\omega_{EF}}$*.*

In the case that the quadrilateral $ABCD$ is orthodiagonal, the following additional properties hold:

Property 6: For any circle ω_i there holds (see [4, Theorem 2]):

(i) *The circle whose diameter is segment* $P_{\omega_i}Q_{\omega_i}$ *(denoted by* σ_i*) intersects side* BC *at points* M_{ω_i} *and* V_{σ_i}*, and side* AD *at points* N_{ω_i} *and* W_{σ_i}*. (It is possible that points* M_{ω_i} *and* V_{σ_i} *or points* N_{ω_i} *and* W_{σ_i} *coincide.)*

(ii) *Segment* $V_{\sigma_i}W_{\sigma_i}$ *is a diameter of circle* σ_i*, therefore quadrilateral* $P_{\omega_i}V_{\sigma_i}Q_{\omega_i}W_{\sigma_i}$ *is a rectangle inscribed in given quadrilateral* $ABCD$*.*

Property 7: For circle ω_{EF}, whose diameter is segment EF, there also holds (see [4, Theorem 3]):

(i) *The circle whose diameter is segment* $P_{\omega_{EF}}Q_{\omega_{EF}}$ *(denoted by* σ_{EF}*) intersects each side of quadrilateral* $ABCD$ *at two points* (see Figure 2b) *as follows: side* AB *at points* $P_{\omega_{EF}}$ *and* $U_{\sigma_{EF}}$*, side* BC *at points* $M_{\omega_{EF}}$ *and* $V_{\sigma_{EF}}$*, side* CD *at points* $Q_{\omega_{EF}}$ *and* $T_{\sigma_{EF}}$*, side* AD *at points* $N_{\omega_{EF}}$ *and* $W_{\sigma_{EF}}$*.*

(ii) *The four chords* $V_{\sigma_{EF}}N_{\omega_{EF}}$*,* $W_{\sigma_{EF}}M_{\omega_{EF}}$*,* $Q_{\omega_{EF}}U_{\sigma_{EF}}$*, and* $P_{\omega_{EF}}T_{\sigma_{EF}}$ *of circle* σ_{EF} *intersect at point* E*.*

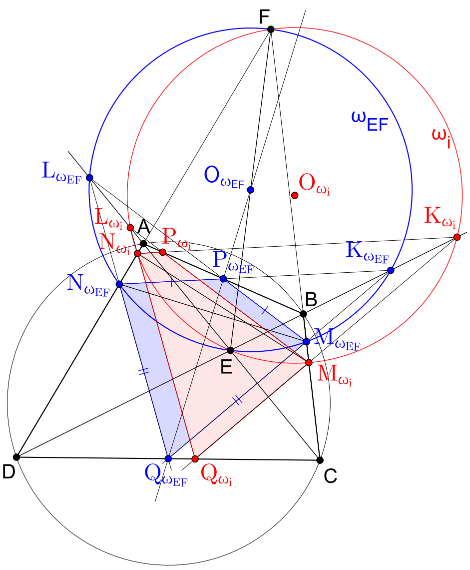

Figure 4

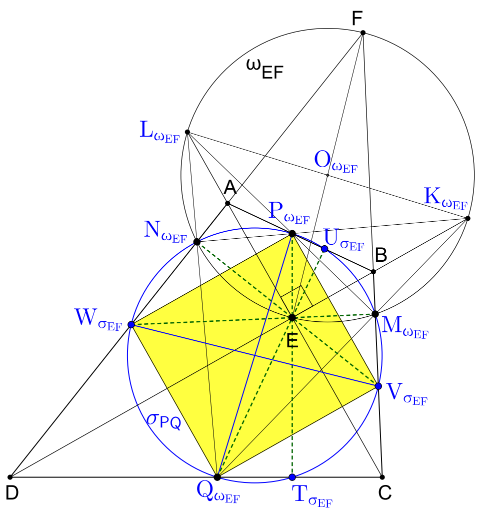

Figure 5

1. Finding all the circles that form Pascal points

The Fundamental Theorem states that for given convex quadrilateral $ABCD$, any circle that passes through the point of intersection of diagonals E; through the points of intersection, F, of the extensions of opposite sides; and through the interior points of sides BC and AD, forms a pair of Pascal points: point P on side AB and point Q on side CD.
The two extreme cases are circle ω_A, which passes through points A, F, and E, and circle ω_B, which passes through points B, F, and E (see Figure 3).

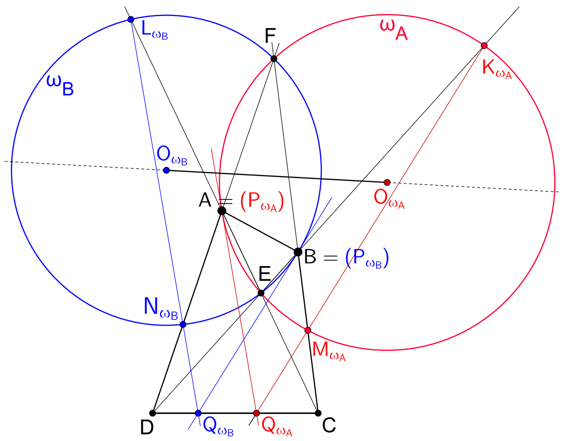

Figure 6

In the case when the circle ω_A passes through an interior point of side BC it forms the following Pascal points: point P_{ω_A}, which coincides with point A, and point Q_{ω_A}, which is an internal point of side CD.
In the case when circle ω_B passes through an interior point of side AD it forms the following Pascal points: point P_{ω_B}, which coincides with point B, and point Q_{ω_B}, which is an internal point of side CD. (see [1]).
In the following theorem, we define the set of all circles that pass through points F and E and through interior points of sides BC and AD.

Theorem 1
Let $ABCD$ be a convex quadrilateral in which E is the point of intersection of diagonals and F is the point of intersection of the extensions of sides BC and AD (in other words, F is the point of intersection of rays CB and DA, (see Figure 1a);

ω_A is a circle that passes through points A, F and E, whose center is at point O_{ω_A};

ω_B is a circle that passes through points B, F and E, whose center is at point O_{ω_B};

ω_A passes through an interior point of the side BC.
Then:

(1) Circle ω_B passes through an interior point of side AD.

(2) *The set of all circles that pass through points E and F and through interior points of sides BC and AD is the set of all the circles that pass through points E and F, and whose centers are located on segment $O_{\omega_A}O_{\omega_B}$.*

Proof.
We first prove the following lemma:

Lemma: *Let $ABCD$ be a convex quadrilateral, and let ω be some circle that passes through points E and F.*
Vertices A and B are interior points of circle ω if and only if ω intersects sides BC and AD at interior points.

Proof of the lemma.
First direction:
Given: $ABCD$ is a convex quadrilateral, ω is some circle that passes through points E and F, and vertices A and B are interior points of circle ω.
We need to prove that circle ω intersects sides BC and AD at interior points.
Proof of the first direction:
Points A, E, and C are located on the same straight line, and point E is located between A and C (which follows from the fact that quadrilateral $ABCF$ is convex).

Circle ω passes through point E, and it is also given that point A is an interior point of the circle, hence it follows that circle ω intersects straight line AC at ray EA and does not intersect ray EC. Point C is an interior point of ray EC and is therefore an exterior point of circle ω.
Since point B is interior to the circle (given) and we have shown that point C is exterior to the circle, then circle ω must intersect side BC.
Similarly, we can prove that point D is exterior to circle ω, and since point A is interior to ω (given), it follows that circle ω intersects side AD.
Second direction:
Given: $ABCD$ is a convex quadrilateral, ω is some circle that passes through points E and F and also intersects sides BC and AD at interior points.
We need to prove that vertices A and B are interior points of circle ω.
Proof of the second direction:
Let us prove an equivalent claim: If vertex A or vertex B are exterior to circle ω, then circle ω does not intersect side BC at an interior point or does not intersect side AD at an interior point.
In the case where point B is exterior to circle ω, we prove that circle ω does not intersect side BC at an interior point.
We shall prove this indirectly by assuming that point C is an interior point of circle ω.
From the data concerning point F, points B and C are located on the ray that issues from point F, and point B is located between F and C, in other words B is an interior point of segment FC.
Circle ω passes through point F. Therefore, if the assumption that point C is interior to circle ω is true, then all the points of segment FC (including point B) are interior points of circle ω, contrary to the given data.
Therefore the assumption is not true, and point C is exterior to circle ω.
Thus, in the case where point B is exterior to circle ω, it follows that ω passes through point F, and that points B and C are exterior to ω and are located on the

ray that issues from point F. Therefore it is not possible for circle ω to intersect segment BC.
In a similar manner we can show that in the case where point A is exterior to circle ω, there holds that point D is exterior to ω, and from similar considerations it follows that circle ω cannot intersect segment AD. □

We return to the proof of the theorem:

(1) It is given that ω_A passes through an interior point of side BC. From the Lemma 1, this is equivalent to point B being an interior point of circle ω_A. We will show that in this case, point A is an interior point of circle ω_B, and therefore, from the Lemma 1, circle ω_B also intersects side AD at an interior point.
Since the centers of all the circles that pass through points E and F are located on the midperpendicular to segment FE, and since circle ω_A passes through points A and E, therefore center O_{ω_A} is the point of intersection of the midperpendicular to segment FE and the midperpendicular to segment AE (denoted by h, see Figure 4).
First, we will prove that if circle ω_A passes through an interior point of side BC, then relative to straight line h, center O_{ω_B} of circle ω_B is located in the half-plane that contains point A.
We assume the contrary: that relative to straight line h, center O_{ω_B}, of circle ω_B is located in the half-plane that does not contain point A.

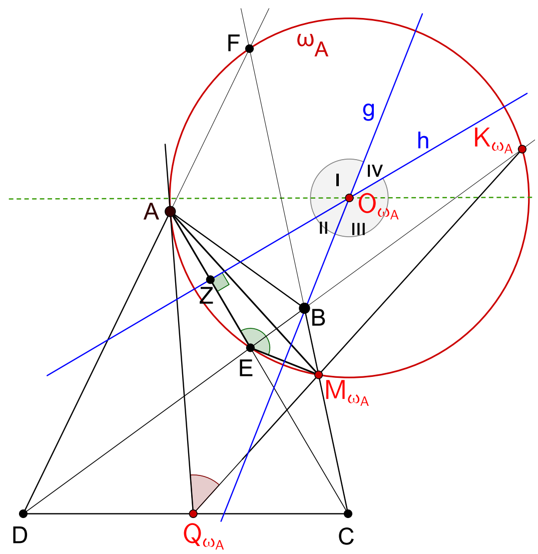

Figure 7

We denote the points of intersection of ω_A with side BC and with the extension of diagonal BD by M_{ω_A} and K_{ω_A}, respectively. We also denote the middle of segment AE by Z (see Figure).
From the Fundamental Theorem, stated above, it follows that the straight line that passes through points M_{ω_A} and K_{ω_A} and the tangent to ω_A at point A intersect at point Q_{ω_A}, which belongs to side CD (see [1, Theorems 1 and 3]).
One can calculate angle $\measuredangle AQ_{\omega_A}K_{\omega_A}$ between secant $Q_{\omega_A}K_{\omega_A}$ and tangent $Q_{\omega_A}A$

as follows: $\measuredangle AQ_{\omega_A}K_{\omega_A} = \frac{1}{2}\left(\widehat{AFK_{\omega_A}} - \widehat{AEM_{\omega_A}}\right)$. Since $\measuredangle AQ_{\omega_A}K_{\omega_A} > 0$, it follows that $\widehat{AFK_{\omega_A}} > \widehat{AEM_{\omega_A}}$.

Therefore, since there holds that: $\widehat{AFK_{\omega_A}} + \widehat{AEM_{\omega_A}} < 360°$, we necessarily have that $\widehat{AEM_{\omega_A}} < 180°$. Therefore, for arc $\widehat{AFM_{\omega_A}}$ (which complements arc $\widehat{AEM_{\omega_A}}$ to form a circle), there holds: $\widehat{AFM_{\omega_A}} = 360° - \widehat{AEM_{\omega_A}} > 180°$.

Angle $\measuredangle AEM_{\omega_A}$ rests on arc $\widehat{AFM_{\omega_A}}$ and therefore $\measuredangle AEM_{\omega_A} = \frac{1}{2}\widehat{AFM_{\omega_A}} > 90°$.

In other words, triangle AEM_{ω_A} is obtuse-angled. Hence the sum of angles $\measuredangle EZO_{\omega_A}$ and $\measuredangle ZEM_{\omega_A}$ is larger than 180°, and therefore the midperpendicular of segment AE (line h) does not intersect side EM_{ω_A} of triangle AEM_{ω_A}.

It follows, therefore, that line h intersects side AM_{ω_A}.

We have obtained that points M_{ω_A} and E are located in the same half-plane relative to straight line h.

Using similar considerations, it can be shown that, relative to the midperpendicular to segment EM_{ω_A} (denoted by g, see Figure 4), points A and E are in the same half-plane.

To summarize, lines h and g divide the plane into four region (Regions (I), (II), (III), and (IV) in Figure 4), where each region is a plane angle whose vertex is at point O_{ω_A}.

Point E is located in a region that is the intersection of the half-plane containing points M_{ω_A} and E relative to line h, with the half-plane containing points A and E relative to line g (region (II) in Figure 4).

Point A is located in a region that is the intersection of the half-plane containing point A relative to line h, with the half-plane containing points A and E relative to line g (region (I) in Figure 4).

Point M_{ω_A} is located in a region that is the intersection of the half-plane containing points M_{ω_A} and E relative to line h, with the half-plane containing point M_{ω_A} relative to line g (region (III) in Figure 4).

Hence, points A and M_{ω_A} are located in different regions whose boundaries are two vertically opposite angles.

The midperpendicular to segment EF passes through point O_{ω_A}, which is the common vertex of all the regions, and therefore it passes through two regions whose boundaries are two vertically opposite angles: one region contains point A and the other region contains point M_{ω_A}.

Based on our contrary assumption, center O_{ω_B} of circle ω_B is located in the half-plane that does not contain point A relative to line h. Therefore O_{ω_B} must be located in Region (III), which contains point M_{ω_A}. In other words, points O_{ω_B} and M_{ω_A} must be located in the same half-plane relative to the midperpendicular to segment EM_{ω_A} (line g).

Hence the distance between O_{ω_B} and M_{ω_A} is smaller than the radius $O_{\omega_B}E$, and therefore M_{ω_A} is an interior point of circle ω_B.

It is given that point B is located on ray FC between points F and C, and it is also given that point M_{ω_A} is an interior point of segment BC. Therefore point M_{ω_A} does not belong to segment FB. Since circle ω_B passes through points F and B, segment FB is a chord of ω_B. Hence it follows that point M_{ω_A} lies outside circle ω_B.

We obtained a contradiction as to the location of point M_{ω_A} relative to circle ω_B.

Therefore the assumption to the contrary is not true, and it thus holds that, relative to the midperpendicular to segment AE (line h), center O_{ω_B} of circle ω_B is located in the half-plane that contains point A.
Hence the distance from O_{ω_B} to A is smaller than radius $O_{\omega_B}E$, and therefore point A is an interior point of circle ω_B.
From the Lemma 1 this is equivalent to circle ω_B intersecting side AD at an interior point.
(2) Let us now consider any circle, ω that passes through points E and F with its center, O_ω, located on segment $O_{\omega_A}O_{\omega_B}$. The center, O_{ω_A}, is located on straight line h, and in Section (1) we proved that relative to line h, center O_{ω_B} is located in the half-plane that contains point A. Therefore, relative to line h, all the interior points of segment $O_{\omega_A}O_{\omega_B}$ (and in particular O_ω) are located in the half-plane that contains point A. Therefore, the distance from O_ω to A is smaller than radius $O_\omega E$, and therefore point A is an interior point of circle ω.
In a similar manner, one can show that, relative to the midperpendicular of segment BE, center O_ω of circle ω is located in the half-plane that contains point B, and therefore point B is an interior point of circle ω.
To summarize, we have shown that in the case where circle ω_A passes through an interior point of side BC, for any circle ω that passes through points E and F, so that its center O_ω is located on segment $O_{\omega_A}O_{\omega_B}$, points A and B are interior to the circle, therefore from the emma 1, circle ω intersects sides BC and AD at interior points.
Now let us consider any circle ω that passes through points E and F so that its center, O_ω, is not located on segment $O_{\omega_A}O_{\omega_B}$. The midperpendicular to segment AE passes through point O_{ω_A}, and the midperpendicular to segment BE passes through point O_{ω_B}. Therefore, center O_ω must be located either in the half-plane that does not contain point A relative to the midperpendicular to segment AE or in the half-plane that does not contain point B relative to the midperpendicular to segment BE.
In the first case, point A is an exterior point of circle ω, and from the emma 1, ω does not intersect side AD at an interior point.
In the second case, point B is an exterior point of circle ω, and from the emma 1, ω does not intersect side BC at an interior point.
In summary, we have obtained that in the case of a convex quadrilateral, $ABCD$, if a circle, ω_A, intersects side BC at an interior point, then the set of all circles that pass through points E and F and through the interior points of sides BC and AD is the set of all circles that pass through points E and F whose center is located on segment $O_{\omega_A}O_{\omega_B}$.

□

2. Definition of a group

In the previous Theorem **1** we defined the set of all the circles that form Pascal points, each circle of this set forms a pair of Pascal points. We therefore defined a set of pairs of Pascal points.
In the following Theorem **2** we shall show that it is possible to define a group over this set.

Definitions.

- (P_ω, Q_ω) is a pair of Pascal points that are formed by circle ω.

- $\mathrm{F}_{AB,CD} = \{(P,Q)|\ P$ and Q are Pascal points on sides AB and CD of quadrilateral $ABCD\ \}$.
  That is to say, $F_{AB,CD}$ is the set of all the pairs of Pascal points on sides AB and CD of quadrilateral $ABCD$.
- $\mathrm{F}'_{AB,CD} = \mathrm{F}_{AB,CD} \setminus \{(P_{\omega_B}, Q_{\omega_B})\}$ is the set of all the pairs of Pascal points on sides AB and CD, aside for the pair formed by circle ω_B.
- $\mathrm{D}_P = |P_{\omega_A} P_{\omega_B}|$ is the length of the segment between the extreme Pascal points P_{ω_A} and P_{ω_B} on side AB of quadrilateral $ABCD$.
- $\mathrm{D}_Q = |Q_{\omega_A} Q_{\omega_B}|$ is the length of the segment between the extreme Pascal points Q_{ω_A} and Q_{ω_B} on side CD of quadrilateral $ABCD$.
- For a pair of Pascal points (P_ω, Q_ω), we define: $\mathrm{d}_{P_\omega} = |P_{\omega_A} P_\omega|$, $\mathrm{d}_{Q_\omega} = |Q_{\omega_A} Q_\omega|$.
  In other words, d_{P_ω} is the distance of Pascal point P_ω from extreme Pascal point P_{ω_A} and d_{Q_ω} is the distance of Pascal point Q_ω from extreme Pascal point Q_{ω_A}.
- Let $ABCD$ be a convex quadrilateral, $(P_{\omega_1}, Q_{\omega_1})$ and $(P_{\omega_2}, Q_{\omega_2})$ - two pairs of Pascal points. The binary operation, $\oplus$, is defined as follows:

$$(P_{\omega_1}, Q_{\omega_1}) \oplus (P_{\omega_2}, Q_{\omega_2}) = \\ = (P_{\omega_3}, Q_{\omega_3}) \quad \left| \begin{array}{l} P_{\omega_3} \in [P_{\omega_A} P_{\omega_B}),\ Q_{\omega_3} \in [Q_{\omega_A} Q_{\omega_B}) \\ \mathrm{d}_{P_{\omega_3}} = \begin{cases} \mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}},\ if\ \mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}} < \mathrm{D}_P \\ \mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}} - \mathrm{D}_P,\ if\ \mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}} \geq \mathrm{D}_P \end{cases} \\ \mathrm{d}_{Q_{\omega_3}} = \begin{cases} \mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}},\ if\ \mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}} < \mathrm{D}_Q \\ \mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}} - \mathrm{D}_Q,\ if\ \mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}} \geq \mathrm{D}_Q \end{cases} \end{array} \right.$$

In other words, the result of the $\oplus$ operation between two pairs of Pascal pairs is a pair of points, $(P_{\omega_3}, Q_{\omega_3})$, where point P_{ω_3} belongs to segment $[P_{\omega_A} P_{\omega_B})$ and the distance of P_{ω_3} from the extreme Pascal point ,P_{ω_A}, is defined as:
In the case where the sum of the distances of points P_{ω_1} and P_{ω_2} from point P_{ω_A} $\left(i.e.\ \mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}}\right)$ is smaller than the length of segment $P_{\omega_A} P_{\omega_B}$, the distance of point P_{ω_3} from point P_{ω_A} is equal to the sum of the distances of points P_{ω_1} and P_{ω_2} from point P_{ω_A}.
In the case where the sum of the distances $\left(\mathrm{d}_{P_{\omega_1}} + \mathrm{d}_{P_{\omega_2}}\right)$, is larger or equal to the length of segment $P_{\omega_A} P_{\omega_B}$, the distance of point P_{ω_3} from point P_{ω_A} is equal to the sum of the distances of points P_{ω_1} and P_{ω_2} from point P_{ω_A} minus the length of segment $P_{\omega_A} P_{\omega_B}$.
Point Q_{ω_3} belongs to segment $[Q_{\omega_A} Q_{\omega_B})$ and the distance of Q_{ω_3} from the extreme Pascal point Q_{ω_A} is defined as follows:
In the case where the sum of the distances of points Q_{ω_1} and Q_{ω_2} from point Q_{ω_A} $\left(i.e.\ \mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}}\right)$ is smaller than the length of segment $Q_{\omega_A} Q_{\omega_B}$, the distance of point Q_{ω_3} from point Q_{ω_A} is equal to the sum of the distances of points Q_{ω_1} and Q_{ω_2} from point Q_{ω_A}.
In the case where the sum of the distances $\left(\mathrm{d}_{Q_{\omega_1}} + \mathrm{d}_{Q_{\omega_2}}\right)$ is larger or equal to the length of segment $Q_{\omega_A} Q_{\omega_B}$, the distance of point Q_{ω_3} from point Q_{ω_A} is equal to the sum of the distances of points Q_{ω_1} and Q_{ω_2} from point Q_{ω_A} minus the length of segment $Q_{\omega_A} Q_{\omega_B}$.

Theorem 2
For any convex quadrilateral, $ABCD$, in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD, if circle, ω_A, that passes through points A, F, and E also passes through an interior point of side BC, then the set $\mathrm{F}'_{AB,CD}$ is a group with respect to the $\oplus$ operation.

Proof.
In Theorem **1** we saw that for a convex quadrilateral, $ABCD$, with a circle, ω_A,

that passes through an interior point of side BC, there holds that the set of all circles that pass through points E and F and through interior points of sides BC and AD is the set of all circles that pass through points E and F and whose centers lie on segment $O_{\omega_A}O_{\omega_B}$.
In this case, the set $\mathrm{F}'_{AB,CD}$ is the set of all pairs of Pascal points formed using the circles that pass through points E and F and whose centers lies on segment $O_{\omega_A}O_{\omega_B}$, aside for the pair formed using circle ω_B.
Let us prove that this set with the operation $\oplus$ satisfies the conditions to be a group.
Proving closure:
Let $(P_{\omega_1}, Q_{\omega_1})$ and $(P_{\omega_2}, Q_{\omega_2})$ be two arbitrary pairs of Pascal points formed by circles ω_1 and ω_2, respectively, and let $(P_{\omega_3}, Q_{\omega_3})$ be a pair of points on sides AB and CD that are obtained using the $\oplus$ operation between the pairs $(P_{\omega_1}, Q_{\omega_1})$ and $(P_{\omega_2}, Q_{\omega_2})$. In other words, there holds: $(P_{\omega_1}, Q_{\omega_1}) \oplus (P_{\omega_2}, Q_{\omega_2}) = (P_{\omega_3}, Q_{\omega_3})$.
We need to prove that $(P_{\omega_3}, Q_{\omega_3}) \in \mathrm{F}'_{AB,CD}$, that is to say, we must show that there exists a circle (which is not ω_B) that passes through points E and F and the Pascal points formed by it are P_{ω_3} and Q_{ω_3}.
We first prove that for the pair of points $(P_{\omega_3}, Q_{\omega_3})$ there holds: $\dfrac{AP_{\omega_3}}{P_{\omega_3}B} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_3}Q_{\omega_B}}$.
The four pairs (A, Q_{ω_A}), (B, Q_{ω_B}), $(P_{\omega_1}, Q_{\omega_1})$, and $(P_{\omega_2}, Q_{\omega_2})$, are pairs of Pascal points formed respectively by the circles ω_A, ω_B, ω_1, and ω_2. Therefore, from property given above, there holds:

$$\frac{AP_{\omega_1}}{P_{\omega_1}B} = \frac{Q_{\omega_A}Q_{\omega_1}}{Q_{\omega_1}Q_{\omega_B}} \text{ and } \frac{AP_{\omega_2}}{P_{\omega_2}B} = \frac{Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_2}Q_{\omega_B}}.$$

From the properties of proportion it also follows that the following proportions hold:

$$\frac{AP_{\omega_1}}{AB} = \frac{Q_{\omega_A}Q_{\omega_1}}{Q_{\omega_A}Q_{\omega_B}} \text{ and } \frac{AP_{\omega_2}}{AB} = \frac{Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_A}Q_{\omega_B}}.$$

We add the last two proportions and obtain:

$$\frac{AP_{\omega_1} + AP_{\omega_2}}{AB} = \frac{Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_A}Q_{\omega_B}}. \tag{2.1}$$

We return to the pair of points $(P_{\omega_3}, Q_{\omega_3})$, which satisfy
$(P_{\omega_1}, Q_{\omega_1}) \oplus (P_{\omega_2}, Q_{\omega_2}) = (P_{\omega_3}, Q_{\omega_3})$.
If the inequality $AP_{\omega_1} + AP_{\omega_2} < AB$ holds, it necessarily follows from (2.1) that the inequality $Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2} < Q_{\omega_A}Q_{\omega_B}$ also holds, and vice versa.
From the definition there holds:

$$AP_{\omega_3} = AP_{\omega_1} + AP_{\omega_2} \text{ and } Q_{\omega_A}Q_{\omega_3} = Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2},$$

and hence from (2.1) it follows that $\dfrac{AP_{\omega_3}}{AB} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_A}Q_{\omega_B}}$.
If the inequality $AP_{\omega_1} + AP_{\omega_2} \geq AB$ holds, it necessarily follows from (2.1) that the inequality $Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2} \geq Q_{\omega_A}Q_{\omega_B}$ also holds, and vice versa.
We subtract 1 from both sides of (2.1), and obtain:

$$\frac{AP_{\omega_1} + AP_{\omega_2}}{AB} - 1 = \frac{Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_A}Q_{\omega_B}} - 1 \Rightarrow$$

$$\Rightarrow \frac{AP_{\omega_1} + AP_{\omega_2} - AB}{AB} = \frac{Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2} - Q_{\omega_A}Q_{\omega_B}}{Q_{\omega_A}Q_{\omega_B}}. \tag{2.2}$$

Now, from the definition there holds:
$AP_{\omega_3} = AP_{\omega_1} + AP_{\omega_2} - AB$ and $Q_{\omega_A}Q_{\omega_3} = Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2} - Q_{\omega_A}Q_{\omega_B}$.
From (2.2) it follows that $\dfrac{AP_{\omega_3}}{AB} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_A}Q_{\omega_B}}$ holds in this case as well.
We have shown that for $(P_{\omega_3}, Q_{\omega_3})$ there holds $\dfrac{AP_{\omega_3}}{AB} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_A}Q_{\omega_B}}$, and therefore, from the properties of proportions there holds: $\dfrac{AP_{\omega_3}}{P_{\omega_3}B} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_3}Q_{\omega_B}} \underset{we\,denote}{=} \alpha$.
Now let us prove that there exists a circle (which differs from ω_B) that passes through points E and F and the Pascal points formed by it are points P_{ω_3} and Q_{ω_3}.
We consider circle ω', which passes through points E and F so that its center, $O_{\omega'}$, is located on segment $O_{\omega_A}O_{\omega_B}$ and divides the segment by the following ratio: $\dfrac{O_{\omega_A}O_{\omega'}}{O_{\omega'}O_{\omega_B}} = \alpha$.
In this case, the following equality holds: $\dfrac{AP_{\omega_3}}{P_{\omega_3}B} = \dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_3}Q_{\omega_B}} = \dfrac{O_{\omega_A}O_{\omega'}}{O_{\omega'}O_{\omega_B}}$.
On the other hand, from Theorem **1** above, circle ω' forms Pascal points, $P_{\omega'}$ and $Q_{\omega'}$, and from property () above, there holds: $\dfrac{AP_{\omega'}}{P_{\omega'}B} = \dfrac{Q_{\omega_A}Q_{\omega'}}{Q_{\omega'}Q_{\omega_B}} = \dfrac{O_{\omega_A}O_{\omega'}}{O_{\omega'}O_{\omega_B}}$.
Therefore there holds that $\dfrac{AP_{\omega_3}}{P_{\omega_3}B} = \dfrac{AP_{\omega'}}{P_{\omega'}B}$ and $\dfrac{Q_{\omega_A}Q_{\omega_3}}{Q_{\omega_3}Q_{\omega_B}} = \dfrac{Q_{\omega_A}Q_{\omega'}}{Q_{\omega'}Q_{\omega_B}}$, and hence point $P_{\omega'}$ coincides with P_{ω_3}, and point $Q_{\omega'}$ coincides with Q_{ω_3}.
Hence it follows that the pair of Pascal points formed by circle ω' is the pair of points $(P_{\omega_3}, Q_{\omega_3})$. □

Proof of the existence of an identity element:

Proof.
For a pair of pascal points (A, Q_{ω_A}) which are formed by the circle ω_A, there holds: $\mathrm{d}_{P_{\omega_A}} = |P_{\omega_A}A| = 0$ and $\mathrm{d}_{Q_{\omega_A}} = |Q_{\omega_A}Q_{\omega_A}| = 0$.
Therefore, for any pair of Pascal points (P_ω, Q_ω), there holds:

$$(P_\omega, Q_\omega) \oplus (A, Q_{\omega_A}) = (P_\omega, Q_\omega).$$

□

Proof of the existence of an inverse element:

Proof.
Let $(P_{\omega_1}, Q_{\omega_1})$ be an arbitrary pair of Pascal points formed by the circle ω_1 (which differs from ω_B). From property () above and the properties of proportions, there holds:

$$\frac{AP_{\omega_1}}{AB} = \frac{Q_{\omega_A}Q_{\omega_1}}{Q_{\omega_A}Q_{\omega_B}} = \frac{O_{\omega_A}O_{\omega_1}}{O_{\omega_A}O_{\omega_B}} \underset{we\,denote}{=} \lambda. \tag{2.3}$$

It is easy to see that $0 \leq \lambda < 1$.
We choose a circle, ω_2, that passes through points E and F, so that its center, O_{ω_2}, is located on segment $O_{\omega_A}O_{\omega_B}$ and divides the segment by the ratio $\dfrac{O_{\omega_A}O_{\omega_2}}{O_{\omega_A}O_{\omega_B}} = 1 - \lambda$.
From property () above and the properties of proportions, center O_{ω_2} and the Pascal

points P_{ω_2} and Q_{ω_2} that are formed by circle ω_2 satisfy the equality:

$$\frac{AP_{\omega_2}}{AB} = \frac{Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_A}Q_{\omega_B}} = \frac{O_{\omega_A}O_{\omega_2}}{O_{\omega_A}O_{\omega_B}}. \tag{2.4}$$

We add equalities (2.3) and (2.4) and obtain:

$$\frac{AP_{\omega_1} + AP_{\omega_2}}{AB} = \frac{Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2}}{Q_{\omega_A}Q_{\omega_B}} = \frac{O_{\omega_A}O_{\omega_1}}{O_{\omega_A}O_{\omega_B}} + \frac{O_{\omega_A}O_{\omega_2}}{O_{\omega_A}O_{\omega_B}} = \lambda + 1 - \lambda = 1.$$

Hence it follows that:
$AP_{\omega_1} + AP_{\omega_2} = AB$ and $Q_{\omega_A}Q_{\omega_1} + Q_{\omega_A}Q_{\omega_2} = Q_{\omega_A}Q_{\omega_B}$.
Therefore, for the pairs $(P_{\omega_1}, Q_{\omega_1})$ and $(P_{\omega_2}, Q_{\omega_2})$ there holds:
$(P_{\omega_1}, Q_{\omega_1}) \oplus (P_{\omega_2}, Q_{\omega_2}) = (P_{\omega_3}, Q_{\omega_3})$, where:
$\mathrm{d}_{P_{\omega_3}} = |P_{\omega_A}P_{\omega_1}| + |P_{\omega_A}P_{\omega_2}| - |P_{\omega_A}P_{\omega_B}| = 0$ and
$\mathrm{d}_{Q_{\omega_3}} = |Q_{\omega_A}Q_{\omega_1}| + |Q_{\omega_A}Q_{\omega_2}| - |Q_{\omega_A}Q_{\omega_B}| = 0$.
In other words, $P_{\omega_3} = A$ and $Q_{\omega_3} = Q_{\omega_A}$.
Therefore there holds: $(P_{\omega_1}, Q_{\omega_1}) \oplus (P_{\omega_2}, Q_{\omega_2}) = (A, Q_{\omega_A})$. In other words, the pair of Pascal points $(P_{\omega_2}, Q_{\omega_2})$ is the inverse of the pair of Pascal points $(P_{\omega_1}, Q_{\omega_1})$. □

Associativity and commutativity are proven by using the associativity and commutativity properties of real numbers and checking all the possibilities for the $\oplus$ operation.

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Department of Computer Science, Technion, Haifa 32000, Israel
E-mail address: dovfraivert@gmail.com

Department of Mathematics, Shaanan College, Haifa 26109, Israel
E-mail address: davidfraivert@gmail.com