CONSTRUCTIONS OF OVALS

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ABSTRACT. Ovals are often used in technical practice to approximate the ellipse. We will show interesting constructions of ovals from authors such as S. Serlio, G. Guarini and F.S. Meyer. We will present new ovals that they are result from the modification of the already known constructions.

INTRODUCTION

Ellipse is an interesting curve for technical practice. But exact construction of ellipse is difficult because its radius of curvature is continually changing. Therefore, they are used of ovals for approximation of ellipse. We can say that oval is a closed curve in a plane which "loosely" resembles the outline of an egg, is differentiable (smooth-looking), convex, simple (not self-intersecting) plane curve, which has at least one axis of symmetry. In a special case and also in architecture, we see the oval as a curve, which is composed of circular arcs. We can find such an approach in books of authors as S. Serlio [1], G. Guarini [2] and F. S. Meyer [6]. They approximate the ellipse by circular arcs. Due to the symmetry of an ellipse it is enough to use two different circles most often. We search the centers and radii of the two circles that are connected smoothly. The circles have a common tangent at the point of connection, which means that the centers H, K (Fig. 1) of these circles must belong on one line. Let $h = |SH|$ and $k = |SK|$ where S is the center of the oval. Found circles have radii $a-h, a-h+\sqrt{h^2+k^2}$ where a is the length of semi-major axis of the ellipse.

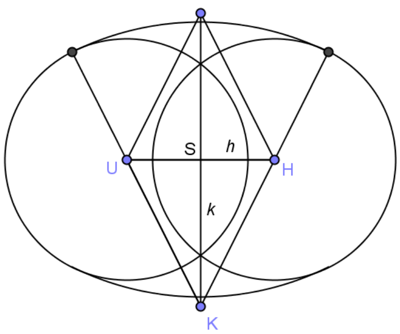

FIGURE 1. Construction of an oval by two circular arcs

We will deal with the already known constructions of ovals from the above mentioned authors. We will modify some constructions so that the circular arcs interpolate of the ellipse in all its vertices, and we will also use a osculating circle in the vertex.

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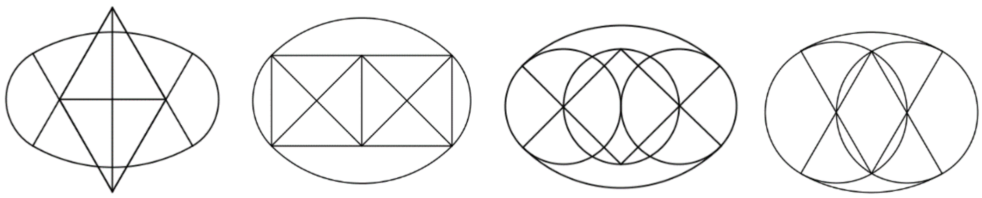
1. OVALS IN TECHNICAL PRACTICE

1.1. Constructions by S. Serlio and G. Guarini

In paper [3], P. L. Rosin described Serlio´s constructions (Fig. 2). The construction of the Fig. 2(a) is for the equilateral triangle ΔUHK and $h = (a - b)(\sqrt{3}-1)$, $k = \sqrt{3}h$. The ratio of the radius of the circles, and the ratio a/b are not constant. These ratios are constant in other Serlio's constructions. In case Fig. 2(b) is $h = k = a/(\sqrt{2}+1)$, $a/b = (\sqrt{2}+1)/(2\sqrt{2}-1)$ and the ratio of the radii of the arcs is 1/2. In case Fig. 2(c) is $h = k = a/2$, $a/b = \sqrt{2}$ and the ratio of the radii of the arcs is $\sqrt{2}-1$. In case Fig. 2(d) is $h = a/3$, $a/b = 3/(4-\sqrt{3})$, $k = \sqrt{3}h$ and the ratio of the radii of the arcs is 1/2.



(a) (b) (c) (d)

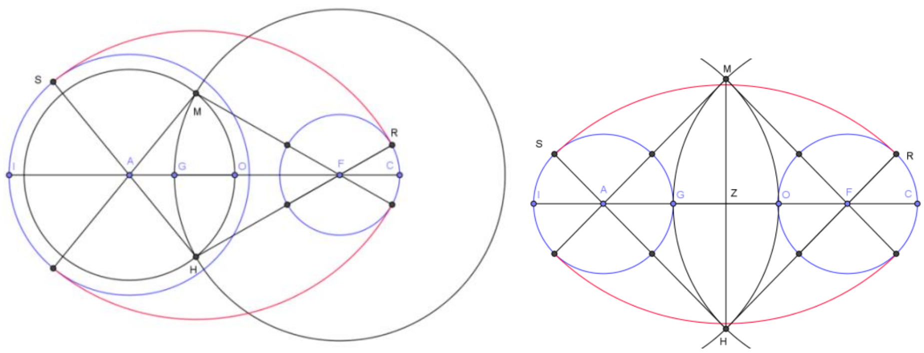
FIGURE 2. Oval constructions by S. Serlio

Guarini (see [2]) presented the general construction of an oval. We have two circles with centers A and F with any distance, with equal or unequal radii. In Fig. 3(a) we use non-intersecting circles with different radii. Line AF intersects these circles in points I and C. We construct points G and O. $|IO|$ is equals $|CG|$ and it is equal to the length greater than half the line segment CI length. We draw circles with centers A and F and radii $|AO|$, $|GF|$. These circles have common the points M and H. The lines HA, HF intersect the circles at points S, R. (Fig. 3), which lie on a circle with center H.

In [4] we proved that the ratio of the radii of the arcs is $2h/r$ and ratio

$$\frac{a}{b} = \frac{h + r}{2h - \sqrt{3h^2 - 4hr + r^2}},$$

where $r = |AI| = |CF|$ and $h = |AF|/2$ (see Fig. 3). We can algebraically expressed the length $k = |ZH|$ as $k = \sqrt{3h^2 - 4hr + r^2}$.



(a) (b)

FIGURE 3. Ovals constructed using Guarini´s method

1.2. Constructions by F. S. Meyer

We have the ellipse given by its axes. F.S. Meyer in paper [6] dealt with constructions of ovals which they interpolate ellipse in all vertices.

Construction 1 [6, Plate 20, pp. 31, no. 11]
Point S is the center of the ellipse. We take half the difference between the semi-major axis and the semi-minor axis of the ellipse. From the center S we apply this distance three times to the major axis and four times to the minor axis. We obtain centers S_1, S_2 of circles that we will use to approximate the ellipse, see Fig. 4.

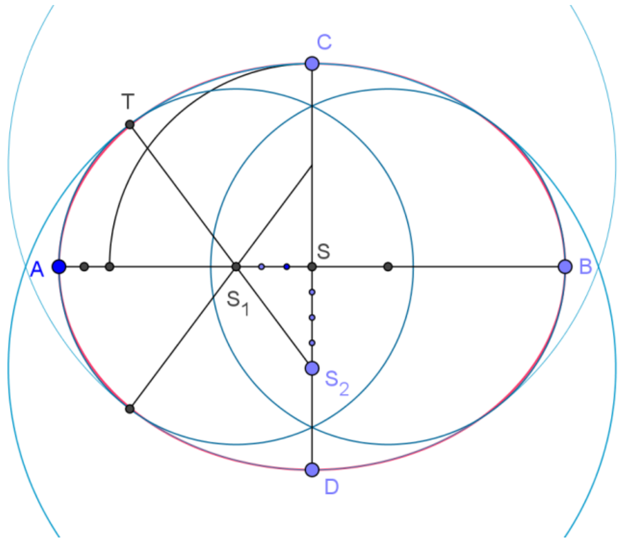


FIGURE 4. Oval constructed using Meyer ´s method

We denote $|SA| = a$, $|SC| = b$ and we have $h = |S_1S| = 3(a\text{-}b)/2$, $k = |SS_2| = 2(a\text{-}b)$, $h = 3k/4$ and radii of circles for centers S_1 and S_2 are $r_1 =$ $(3b\text{-}a)/2$ and $r_2 = 2a\text{-}b$. Based on the construction is $|S_1S_2| = 5(a\text{-}b)/2$. $|S_2T| = |S_1S_2|+|S_1T| = 5(a\text{-}b)/2 + (3b\text{-}a)/2 = 2a\text{-}b = |S_2C|$. Ellipse (red colour) in Fig. 4 has $a/b = 5/4$.

Construction 2 [6, Plate 20, pp. 31, no. 10]
We construct (Fig. 5) a line segment AC and apply the length $a\text{-}b$ from the point C to this line segment and we get a point Q. We denote the center of the line segment AQ as a point L. We draw the perpendicular to the line AC through the point L. This line intersects the major axis at a point S_1 and the minor axis at a point S_2, which are centers of circles that approximate these ellipse. Based on the construction, the triangles $\Delta\, ALS_1$, $\Delta\, S_2LC$, $\Delta\, ASC$, $\Delta\, S_2SS_1$ are similar. In [7] we proved that the lengths $h = |SS_1| = a - |AS_1| = \frac{(a-b)\left(\sqrt{a^2+b^2}+a+b\right)}{2a}$, $k = |S_2S| = a|SS_1|/b = ah/b$, $|AL| = \frac{\sqrt{a^2+b^2}-(a-b)}{2}$ and radii of circles for centers S_1 and S_2 are

$$r_1 = |AS_1| = \frac{|AL||AC|}{|AS|} = \frac{\sqrt{a^2+b^2}(\sqrt{a^2+b^2}-(a-b))}{2a},$$

$$r_2 = |CS_2| = \frac{|CL||AC|}{|CS|} = \frac{\sqrt{a^2+b^2}(\sqrt{a^2+b^2}+(a-b))}{2b}.$$

In Fig. 5 is the ellipse with $a/b = 5/4$ (red colour) and its approximation according to the construction just described.

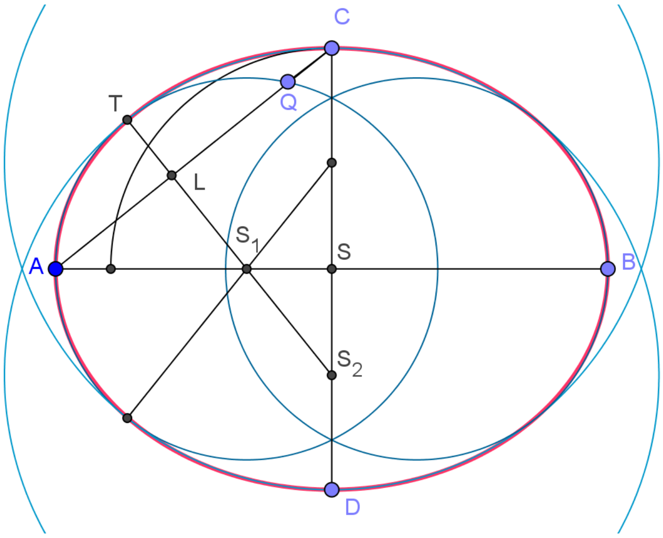


FIGURE 5. Oval constructed using Meyer ´s method

Construction 3 [6, Plate 20, pp. 31, no. 12]
F. S. Meyer showed the construction where he used the osculating circles of the ellipse and the ellipse is approximated by an oval determined by three different circles (see Fig. 6).

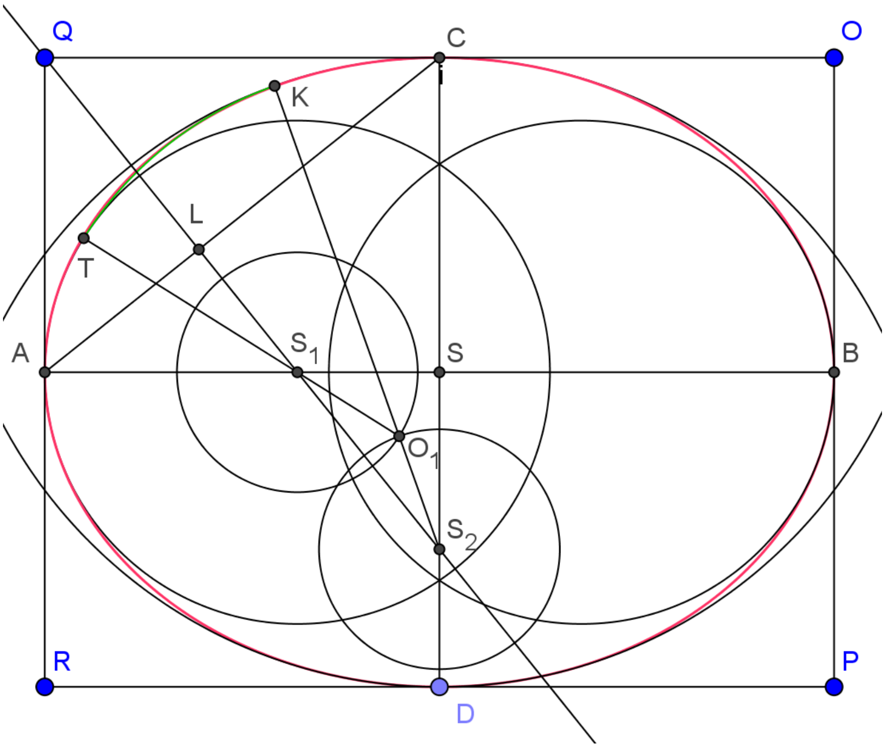


FIGURE 6. Oval constructed using Meyer ´s method

In Fig. 6 for osculating circles in vertices of ellipse we use known construction of centers as Meyer. We construct a rectangle *ORPQ,* the axes of its sides are line segments *AB*, *CD*.

We draw a line segment AC. In the point Q draw a perpendicular line to the line AC. This line crosses the major axis in point S_1 and the minor axis in point S_2, which are the centers of osculating circles.
The radius of the osculating circles is $r_1 = |AS_1| = b^2/a$, $r_2 = |CS_2| = a^2/b$. The similarity of triangles ΔALS_1, ΔS_2LC, ΔASC, ΔS_2SS_1 $\Delta ASC \approx \Delta S_2SS_1$ gives $k = |S_2S| = a|SS_1|/b = ah/b$, where $h = a - r_1 = (a^2 - b^2)/a$.
We draw circles with centers at points S_1, S_2 and with the radius $(r_2 - r_1)/2$. In their intersection is the center O_1 of the circle (green colour, Fig. 6), which smoothly feeds the osculating circles into the continuous curve - oval. Ellipse (red colour) in Fig. 6 has $a/b = 5/4$ and its approximation by three circular arcs.

2. NEW OVALS

When we want to construct an oval which approximate the ellipse with known the major and the minor axes, then the following equality must be satisfied

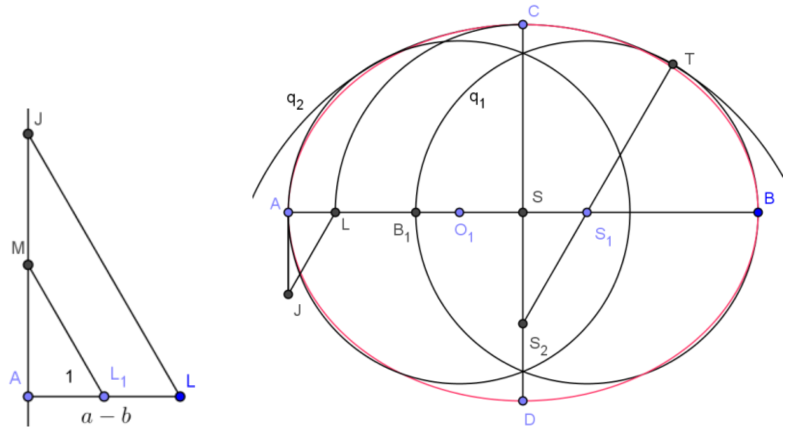
$$r_1 + \sqrt{h^2 + k^2} = k + b, \tag{1}$$

where $h = a - r_1$ and r_1 is a radius of a circle q_1, which approximates the ellipse in its vertex. If we calculate unknown k from the equation (1) we have

$$k = \frac{(a-b)}{2}\frac{(a+b-2r_1)}{b-r_1} \tag{2}$$

2.1. New ovals as modification of Serlio´s constructions

We know for ellipse the lengths of the semi-major and the semi-minor axes are a, b. In [7] we showed for the Serlio´s construction in Fig. 2(a) that $k = (a-b)(3+\sqrt{3})/2$ and $r_1 = (a+b-\sqrt{3}(a-b))/2$.
We construct a line segment of length $\sqrt{3}(a-b)$ as in Fig. 7(a), where $|AL| = a - b$, $|AL_1| = 1$, $|ML_1| = 2$ and $|AJ| = \sqrt{3}(a-b)$. We presented in Fig. 7 (b) the construction of the ellipse whose $a/b = 5/4$ (red colour). The point S_1 is the center of the line segment BB_1, where $|LB_1| = |AJ| = \sqrt{3}(a-b)$, $|LB| = a + b$, $|AL| = a - b = 1$.



(a) (b)

FIGURE 7. Modification of Serlio´s construction (a)

If we take another ratio a/b as shown in Serlio´s constructions (b), (c), (d), these constructions cannot be used for interpolating all vertices of the given ellipse. That is why we modify them. Take the Serlio´s construction in Fig. 2(b). We know that $h = a/(1+\sqrt{2})$ and thus $r_1 = a - h = a(2 - \sqrt{2})$. From (2) it follows that $k = \frac{(a-b)}{2}\frac{(a+b-2a(2-\sqrt{2}))}{b-a(2-\sqrt{2})}$.

In the Fig. 8 we take the ellipse $a = 5$, $b = 4$, we calculate that $k = (34 - 5\sqrt{2})/28$ and we can draw the searched circular arcs (green colour). The circular arc drawn by Serlio has a blue colour.

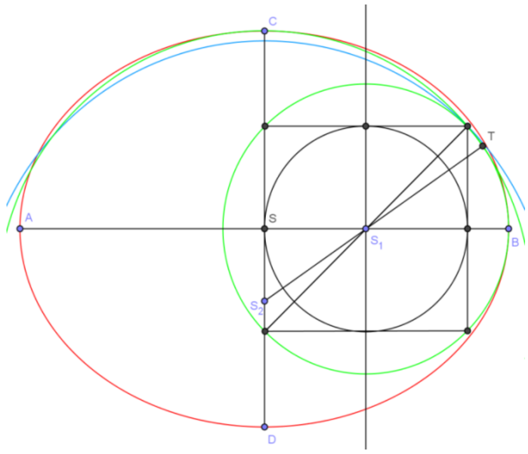


FIGURE 8. Modification of Serlio´s construction (b)

For the Serlio´s construction in Fig. 2(c) is $h = r_1 = a/2$. From (2) show that $k = \frac{b(a-b)}{2b-a}$. In this case, we can a length k to determine constructively. The length $a - b = |AJ| = |SM|$, $2b - a = |LK| = |SN|$. From the similarity of triangles ΔNSD, $\Delta\ MSS_2$, it is obvious that $|SS_2|/b = (a - b)/(2b - a)$ where $k = |SS_2|$. In Fig. 9 (a/b is 5/4) the green colour is for searched circular arcs and blue colour, there is a circular arc constructed according to Serlio.

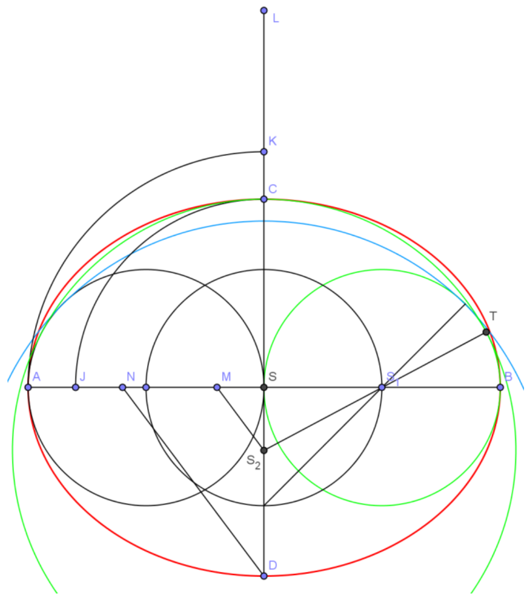


FIGURE 9. Modification of Serlio´s construction (c)

We take the construction in Fig. 2(d) according to Serlio. We know that $h = a/3$, $r_1 = 2a/3$. From (2) it follows that $k = \frac{(a-b)}{2}\frac{(b-a/3)}{b-2a/3}$.
If we take in the Fig. 10 the ellipse where $a = 5$, $b = 4$, we can calculate that $k = 7/4$ and we can draw the searched circular arcs (green colour). The circular arc drawn by Serlio has a blue colour.

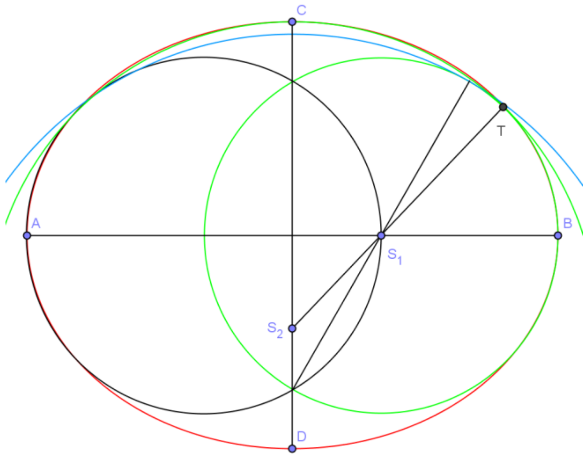


FIGURE 10. Modification of Serlio´s construction (d)

We see that these new modifications give us a nice approximation of the given ellipse. We take addition to the previous condition another condition that the circular arc for a vertex of ellipse is the osculating circle of the ellipse for this vertex. We know that the radius of this circle is $r_1 = b^2/a$.
In Serlio´s construction (a) is $r_1 = (a + b - \sqrt{3}(a - b))/2 = b^2/a$. After the editing we have that

$$1-\sqrt{3}+\frac{b}{a}\left(1+\sqrt{3}\right)-2\frac{b^2}{a^2}=0.$$

From this condition is $\frac{b}{a} = \frac{1+\sqrt{3}\pm\sqrt{12-6\sqrt{3}}}{4}$. It is obvious that for ellipses with this ratio b/a the circle arc q_1 approximating the ellipse around the vertex, it is also an osculating circle for this ellipse in its vertex.

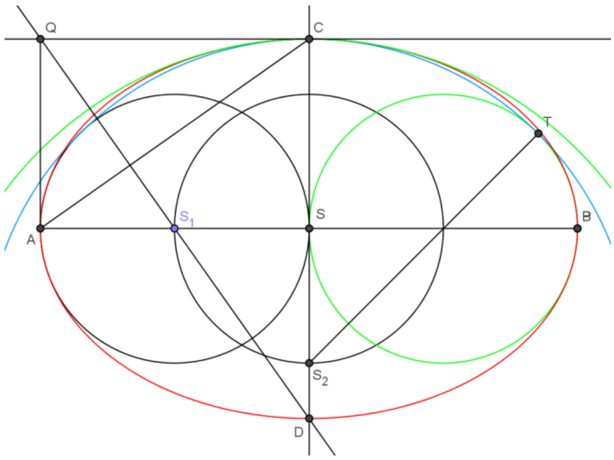


FIGURE 11. Serlio´s construction (c)

For Serlio´s construction (c) is $r_1 = a/2 = b^2/a$ and thus is $a^2/b^2 = 2$. It follows that in this case, in the Serlio´s construction, the osculating circle of the ellipse is chosen. The ellipse is given by lengths a,b for which $a/b = \sqrt{2}$. We constructed in Fig. 11 the circle according to S. Serlio with a blue colour, and for comparison, the osculating circle for the co-vertex C with green colour.
For Serlio´s constructions (b) and (d), there is no ellipse for which the circle arc for the vertex is also the osculating circle of the given ellipse in this vertex.
If we take Serlio´s construction (b), it should be satisfy that $r_1 = a(2-\sqrt{2}) = b^2/a$. It is clear that $a^2/b^2 = 1/(2-\sqrt{2})$. However, for this construction is $a/b = (\sqrt{2}+1)/(2\sqrt{2}-1)$. Similarly for the construction (d) of Serlio is $r_1 = 2a/3 = b^2/a$. Of this $a^2/b^2 = 3/2$. However, only $a/b = 3/(4-\sqrt{3})$ is possible for this type of construction.

2.2. New ovals as modification of Guarini´s constructions

We dealt with the modification of the Guarini´s construction in [7]. Let the circle q_1 be the osculating circle for the point A with the center at point S_1 (Fig. 12). For its radius $r_1 = b^2/a$ and the equation (2) shows that

$$|SS_2| = |SH| = k = \frac{(a-b)}{2}\frac{(a+2b)}{b}.$$

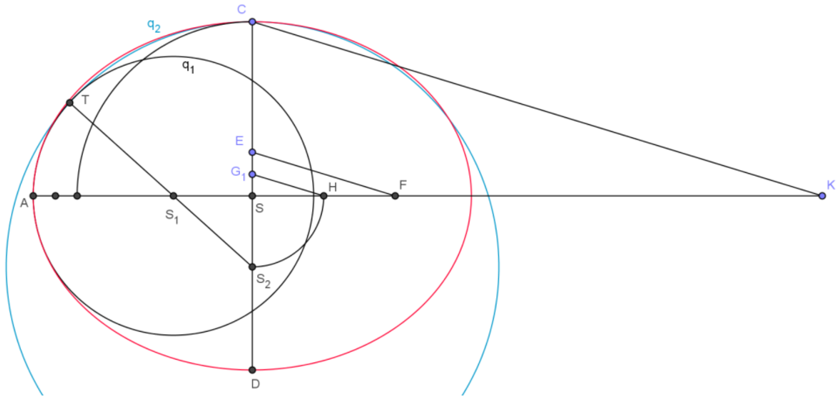


FIGURE 12. Modification of Guarini´s construction

The point S_2 is the center of the circle q_2 which approximates the ellipse in the co-vertex C. Construction of line segment SH is based on the similarity of the triangles $\Delta\ SHG_1$, $\Delta\ SFC$, $\Delta\ SKC$, where $|SG_1| = (a-b)/2$, $|SE| = 1$, $|SK| = a + 2b$, $|SF| = (a+2b)/b$.

3. COMPARISON OF CONSTRUCTIONS

We now compare our modified Guarini's construction with Meyer´s construction. It is clear that Meyer's construction is more precise (Fig. 13 and Fig. 14), because we use both osculating circles of the ellipse (green colour). In modified Guarini's construction we use the circle q_2 (blue colour in Fig. 13 or Fig. 14) which is not the osculating circle of the ellipse (red colour).

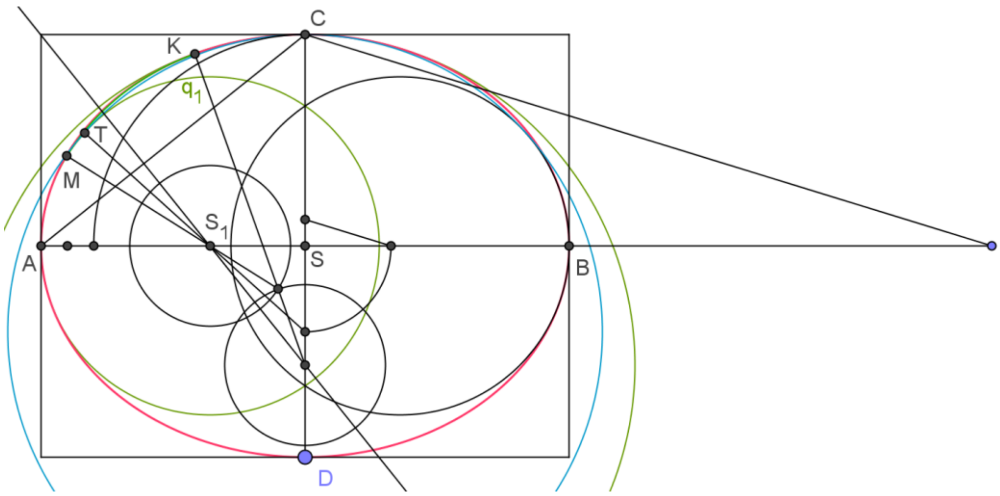


FIGURE 13. Guarini and Meyer construction

Point *T* is the tangent point of the circles in the Guarini´s construction (Fig. 14), and points *M*, *K* are tangent points of the osculating arcs with a third circular arc in the Meyer´s construction.

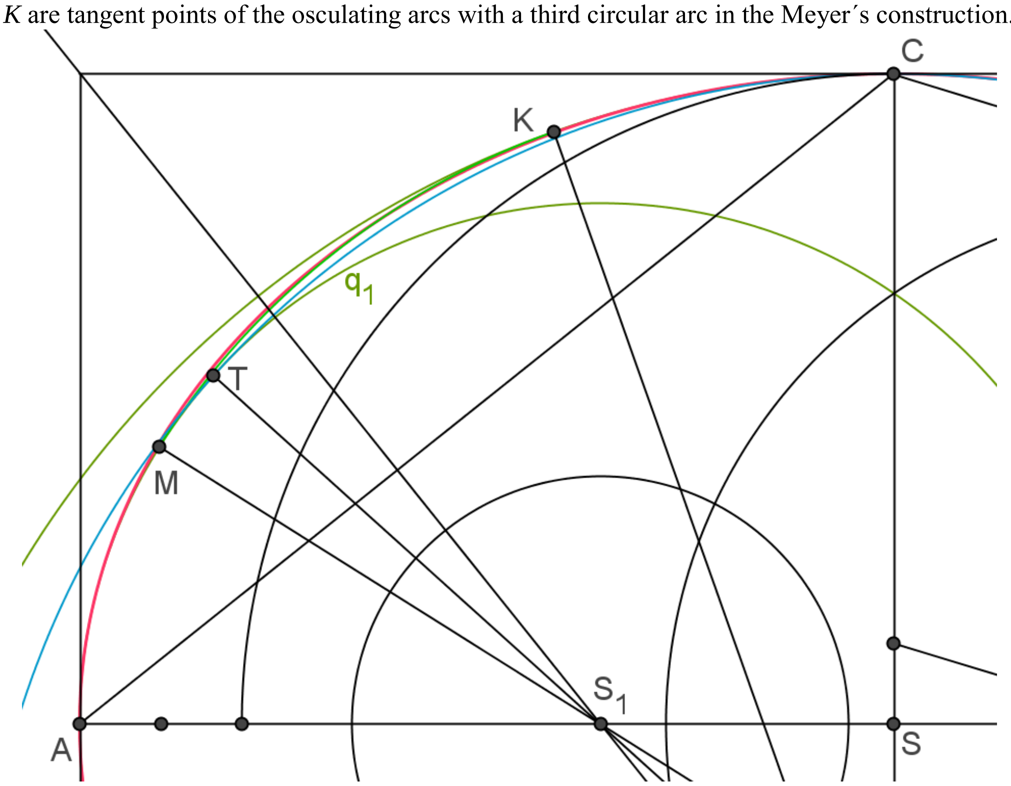


FIGURE 14. Guarini and Meyer construction

We can say that given ellipses are the most precisely approximated by the Meyer´s constructions because he used osculating circles.

The modified Serlio´s and Guarini´s constructions do not give us more precise approximation of the ellipse than the Meyer´s construction in Fig. 6. However, we used only two circular arcs.

CONCLUSION

The difference between the ovals presented in this article and the ellipse is minimal. Therefore, in technical practice, we think that they were used ovals for approximation of the ellipse. For interpreting an elliptical shape, these constructions are, in our opinion, sufficient and practically usable. We do not have the knowledge that new ovals presented in this paper would have been used in the past even though they give us interesting approximations of the ellipse. We believe that these modifications which we presented, will find application in the future.

REFERENCES

[1] SERLIO, S., Il primo [-secondo] libro d'architettura. Venice, 1545. (italian) https://archive.org/details/gri_33125008448702.

[2] GUARINI, G., GUJENOTTI, F., ABBIATI, G., Architettura civile: opera postuma. Published 1737. https://archive.org/details/gri_33125008641249.

[3] ROSIN, P. L., On Serlio's constructions of ovals. The Mathematical Intelligencer, 23(1), 2001, pp. 58-69. DOI: 10.1007/BF03024523.

[4] GRÚŇOVÁ, Z., HOLEŠOVÁ, M., Ellipse and oval in baroque sacral architecture in Slovakia. Civil and environmental engineering. Vol. 13, No. 1, 2017, pp. 30-41.

[5] GRÚŇOVÁ, Z., HOLEŠOVÁ, M., Geometrical analysis of Palesch family chapel in Kľačno, former Gaidel in Western Slovakia. RSP 2017 – XXVI R-S-P Seminar 2017 Theoretical Foundation of Civil Engineering, Vol. 117, No. 00060, 2017.

[6] MEYER, F. S., A Hand Book of Ornament. Published 1896. https://archive.org/details/handbookoforname00meyeuoft.

[7] HOLEŠOVÁ, M., Serlio's, Guarini´s and Meyer´s constructions of ovals in architecture. Proceedings, 17th conference on Applied mathematics, Aplimat 2018, February 6-8, 2018, pp. 450-459

[8] HOLEŠOVÁ, M., Interesting geometric constructions in architecture and their modifications. Proceedings of Slovak-Czech conference on geometry and graphics. September 10-13, 2018, pp. 91-98.

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