ABSOLUTE CIRCLE (SPHERE) GEOMETRY BY REFLECTION

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ABSTRAKT. In this survey we refresh and slightly extend the classical circle geometry and circle inversion as basic transformation, so that they can be extended to non-Euclidean planes, mainly to the Bolyai-Lobachevsky hyperbolic plane. These involve analytic discussions and extension to higher dimensions, based on projective polarity (metric), and lead to generalized Poincaré (conformal) models of non-Euclidean geometries. Novelties arise that we reformulate and extend (slightly renew) our German papers, and "introduce" them into the English electronic literature. We mainly concentrate on plane geometries, higher dimensional cases cause "only technical" (but tiresome) difficulties.

INTRODUCTION

Our paper is an extended version of our conference paper [10], presented also at CSCGG 2018, Blansko, Czech Republic and – briefly – in its Proceedings.

The modern *absolute plane geometry* (extended from the original sense of János Bolyai, 1831) is based on the concept of *projective metric plane* $\mathbf{P}^2(\mathbf{K}, \mathbf{V}^3, \boldsymbol{V}_3, \sim, \langle,\rangle)$ over a (say left) $\mathbf{K}$ *vector space* $\mathbf{V}^3$ and its (then right) *dual linear form space* $\boldsymbol{V}_3$ after Felix Klein's *Erlangen program* (1871). Characteristic of the *basic field Char* $\mathbf{K} \neq 2$ will be a natural assumption. Vectors $\mathbf{X} \sim c\mathbf{X} \in \mathbf{V}^3$ $(0 \neq c \in \mathbf{K})$ describe the same point X, similarly forms $\boldsymbol{u} \sim \boldsymbol{u}c \in \boldsymbol{V}_3$ describe the same line u. $X \in u$ (or X I u) means $(\mathbf{X}\,\boldsymbol{u}) = 0$ (the linear form $\boldsymbol{u}$ take zero $(0 \in \mathbf{K})$ on the vector $\mathbf{X}$), describing the *point-line incidence.*

Nowadays we can speak on *9 real projective metric planes (so-called Cayley–Klein planes).* These $9 = 3^2$ are on the base of the 3 possible one-dimensional projective mappings: *i*) *elliptic* or fixed-point-free; *ii*) *parabolic* or of 1 fixed point; *iii*) *hyperbolic* or of 2 fixed points. These induce the possible projective metrics. First, for distance between point pair in any line; second – dually – 3 possible angular metrics between line pair in any line pencil (see [5], [16], [12], [13]). Namely, we get: the elliptic plane (if we identify the opposite points of the sphere, so more visually the *spherical plane*; both) denoted by $\mathbf{S}^2$ (for simplicity, in projective sense it is self-dual); the *Euclidean plane* $\mathbf{E}^2$ with parabolic distance metric and elliptic angular one, and its dual $\mathbf{DE}^2$. Then come *hyperbolic* (or Bolyai-Lobachevsky) *plane* $\mathbf{H}^2$ with hyperbolic distance metric, elliptic angular one; and its dual $\mathbf{DH}^2$; *Minkowski* (called also Lorentz) *plane* $\mathbf{M}^2$, and its dual $\mathbf{DM}^2$; *Galilei (or isotropic) plane* $\mathbf{G}^2$, it is self-dual. Please, think about and continue (*bihyperbolic plane* is the only difficult case [12], [13])! There are 6 *non-isomorphic plane possibilities*, if duals are not distinguished (in some sense 5 ones are also acceptable, because bihyperbolic plane seems to be too artificial, and the projective extension naturally distinguishes only the extended $\mathbf{PH}^2$ and its dual).

Dual planes occur, if we change the roles of points and lines; or – analytically – we change the describing vector space $\mathbf{V}^3$ and its linear dual form space $\boldsymbol{V}_3$, now over the *real coordinate field* $(\mathbf{R} = \mathbf{K})$. Finite fields also provide a schematic generalization [4].

2010 *Mathematics Subject Classification.* 51F05, 51M15, 51M09.

Key words and phrases. reflection, circle, cycle, inversion, absolute projective metric plane, extension to higher dimensions.

Thus, other points of view come also into considerations in the following! So we shall consider only the generalized circle geometry of $\mathbf{E}^2$, $\mathbf{S}^2$ ***and*** $\mathbf{H}^2$ ***in this paper (although*** $\mathbf{M}^2$ ***would also be involved into our discussion. But*** $\mathbf{G}^2$ ***will be naturally excluded, although circle geometry in*** $\mathbf{G}^2$ ***can also be introduced in another way [13], [14]).***

The above *scalar product* $\langle , \rangle$ can also be derived by a symmetric linear mapping, or *projective metric polarity*: Either, mapping π from line (dual vector = ***form***) to its (pole) point (pole **vector**), as follows in our later notations

π: $\boldsymbol{V}_3 \to \mathbf{V}^3$, $x(\boldsymbol{x}) \to x_*(\boldsymbol{x}_*) := X(\mathbf{X})$, $x_i\boldsymbol{e}^i \to x_i\boldsymbol{e}_*^i ::= x_i\pi^{ij}\mathbf{E}_j =: X^j\mathbf{E}_j$, where $\pi^{ij} = \pi^{ji}$

means *symmetry*. This equivalently involves a symmetric bilinear form (scalar product) on $\boldsymbol{V}_3$

$$\langle , \rangle: \boldsymbol{V}_3 \times \boldsymbol{V}_3 \to \mathbf{K},\ \langle \boldsymbol{x}, \boldsymbol{y} \rangle = (\boldsymbol{x}_*\boldsymbol{y}) = (\mathbf{X}\boldsymbol{y}) = x_i\pi^{ij}y_j = \langle \boldsymbol{y}, \boldsymbol{x} \rangle, \tag{1.1}$$

with Einstein-Schouten sum index conventions for dual basis pair: $\{\mathbf{E}_i\}$ and $\{\boldsymbol{e}^j\}$ with $(\mathbf{E}_i\boldsymbol{e}^j) = \delta_i^j$ the Kronecker symbol; this is the (general) case for classical planes. Or, mapping Π from point (**vector**) to its polar line (polar ***form***), will be introduced as follows

Π: $\mathbf{V}^3 \to \boldsymbol{V}_3$, $X(\mathbf{X}) \to X^*(\mathbf{X}^*) := x(\boldsymbol{x})$, $X^i\mathbf{E}_i \to \mathbf{E}_i^* X^i := \boldsymbol{e}^j\Pi_{ji}X^i := \boldsymbol{e}^j x_j$, where $\Pi_{ji} = \Pi_{ij}$

(again for symmetry). This is also equivalent to a symmetric scalar product on $\mathbf{V}^3$

$$\langle , \rangle: \mathbf{V}^3 \times \mathbf{V}^3 \to \mathbf{K},\ \langle \mathbf{X}, \mathbf{Y} \rangle = (\mathbf{X}\,\mathbf{Y}^*) = X^i\Pi_{ij}Y^j = \langle \mathbf{Y}, \mathbf{X} \rangle \tag{1.2}$$

These come (in general) for dual geometries. If π is bijective (invertible), so is Π as well and $(\pi^{ij})^{-1} = \Pi_{ij}$, of course. This situation occurs for $\mathbf{S}^2$, if π and so Π are positive definite (or of signature (+ + +)), for the dual pair $\mathbf{H}^2$ (+ + -) and $\mathbf{DH}^2$ (- - +). For the remaining dual geometries, π *and* Π are degenerate, respectively. E.g. for $\mathbf{E}^2$ (+ + 0) parallel lines have the same pole, and all poles lie in the so-called ideal line i. On i we shall have a fixed-point-free pole involution, for defining orthogonality of lines, as usual. The pole of i will be the symbol ∞, the unique infinity point of the Euclidean type circle geometry (corresponding to the zero vector $\mathbf{0}$, excluded in general). *Please imagine then* $\mathbf{DE}^2$. We shall have similar situation in Minkowski plane $\mathbf{M}^2$, again parallel lines have one pole on the ideal line i. But here we shall have incident pole and polar line, determining so-called hyperbolic involution on i with two fixed points, even assigning the so-called light directions.

Galilei plane $\mathbf{G}^2$ shall have a unique extra pole I for all common lines, and a unique extra polar line i for all common points, *where* $I \in i$. *The Galilei scalar product* $\langle , \rangle$ will be specific, not detailed here.

The physical interpretations with *Galilei-Newton mechanics* (line-time) for $\mathbf{G}^2$, and with *the mechanics of special relativity (lin*e-time) for $\mathbf{M}^2$, respectively, have important applications in space-time geometry, i.e. in four dimensions.

The corresponding circle geometries will be our topic. The classical Euclidean case $\mathbf{E}^2$ is well-known. The line reflection will be our main transformation for generalization, as nicely elaborated by Friedrich Bachmann in [1]. This leads us to the generalized circle concept, as to cycles (circle, horocycle, hypercycle in $\mathbf{H}^2$, as discussed in [6 – 8]), moreover to the generalized inversion as cycle preserving transformation. The other Cayley-Klein geometries are also well elaborated, but in another way. I.M. Yaglom has nicely reported the classical results in [13 – 15]. My supervisor, Professor Julius (Gyula) Strommer wrote a dissertation [12] on connections with geometric constructions (e.g. exclusively with a compass), see also my obituary [9].

Of course, there are a lot of newer investigations in different directions, see e.g. Benz [2], Juhász [4] and Onishchik – Sulanke [11]. ***I shall be curious for Reader's opinion, related to the present discussion.***

For simplicity (and for the sake of brevity) the discussion will be restricted here, mainly for generalized Euclidean and hyperbolic circle geometry.

In the so-called *reflection geometry* [1] *the axioms of three line reflections* (see Sect. 2 and Fig. 3) play basic roles. These are very strong axioms, mainly involving (together with the other axioms, of course) that a metric plane can be embedded to an above projective metric plane $\mathbf{P}^2(\mathbf{K}, \mathbf{V}^3, \boldsymbol{V}_3, \sim, \langle,\rangle)$ with a specified scalar product $\langle,\rangle$. In a projective plane $\mathbf{P}^2$ with *Char* $\mathbf{K} \neq 2$ there is a unique *harmonic homology* σ as its involutive (involutory) collineation, determined by any quadrangle A, A^σ, B, B^σ (in general position, i.e. no three of them lie in a line) and its three diagonal points $U = AA^\sigma \cap BB^\sigma$, $V = AB \cap A^\sigma B^\sigma$, $W = AB^\sigma \cap A^\sigma B$. U will be centre, VW will be axis of the homology (as *central axial collineation*), $U \notin VW =: u$ is a consequence of *Char* $\mathbf{K} \neq 2$ (Fig. 1). Then the homology or reflection σ will be denoted also by σ_U^u, and called preferably line reflection $\mathbf{u}$, or point reflection $\boldsymbol{U}$, equivalently.

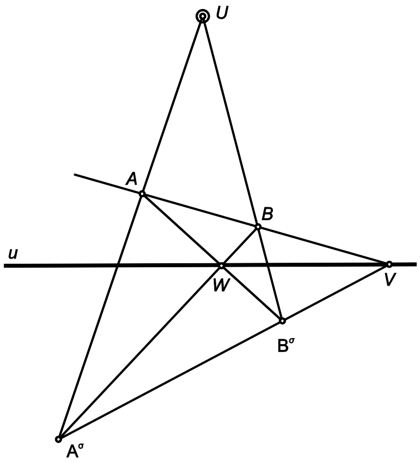


Fig. 1. Projective reflection σ as harmonic homology with centre U and axis u

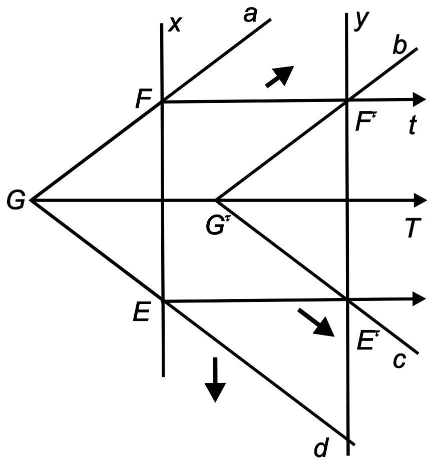


Fig. 2. Projective translation τ with incident *centre T and axis t, the ideal line here*

1.1. A preliminary sketch of cycle and inversion as cycle preserving mapping of a classical projective metric (Euclidean or hyperbolic) plane

Definition 1.1. It is natural to define a *cycle* $\mathcal{C}(A, A_0)$ (the generalized circle) as reflection images A_i of a starting point A_0 in all the lines a_i through the generalized centre A. Symbolically:

$$\mathcal{C}(A, A_0) := \{A_i := A_0^{a_i} \mid a_i \ni A\}, \text{ Fig. 4,} \tag{1.3}$$

also with hyperbolic *circle*, *hypercycle* and *horocycle* (A is *inner*, *outer* and *boundary point*, respectively; A_0 can also be varied). *Line cycle* will consist of the touching lines of a point cycle. □

The *general inversion,* as a cycle preserving point mapping will be defined in more steps. Here consider only Fig. 12-13 for some (Euclidean and hyperbolic) cases.

Definition 1.2. *Inversion* $\alpha(T, A_1, B_1)$ is a mapping of the plane – more precisely, on a point class of moveable into each other points – determined by a point A_1 and its image B_1 and the inversion centre T, where they are different points of a line t_1. Let α order to a point A_i - not on t_1 – a point B_i on the line $t_i = TA_i$, so that A_1, B_1, A_i, B_i lie on a cycle $\mathcal{C}(X_{1i}, A_1)$. Its centre X_{1i} is determined by the symmetry line x_i of A_iB_i and the symmetry line a_{1i} of A_1A_i. The *three reflections equation*

$$b_{1i} := x_1 a_{1i} x_i \text{ will ensure that } B_1^{b_{1i}} := B_1^{x_1 a_{1i} x_i} = A_1^{a_{1i} x_i} = A_i^{x_i} = B_i. \tag{1.4}$$

Consequently, this inversion will be an involutive mapping (it differs of the usual cycle reflection that is only a specific case!). □

Finally, all these discussions can be extended to one dimension higher for plane (hyperplane) intersections of *generalized spheres* (*quadric or quadratic*) and their reflections, as extended inversions, again by harmonic homologies, in one dimension more, onto $\mathbf{P}^3 = \mathbf{P}^{n+1}$ (Fig. 14-16).

1.2. A sketch of F. Bachmann's metric plane for taste and motivation

Metric group plane $\mathcal{M} := \mathcal{M}(\mathcal{G},\mathcal{S})$ is an axiomatically defined structure (*in a group language*, as a novelty), where $\mathcal{G}(\mathcal{S})$ is an involutively generated group by the system $\mathcal{S}$ invariant in $\mathcal{G}$.

$\alpha|\beta$ means that α, β, $\alpha\beta$ of $\mathcal{G}$ are involutive (involutory), $\nmid$ denotes the negation of $|$. Let $\mathcal{P}$ be the set of all involutive products of elements from $\mathcal{S}$. We can define a geometric structure $\mathcal{M} := \mathcal{M}(\mathcal{G},\mathcal{S})$: Let the elements of $\mathcal{S}$ call lines and denoted by *a*, *b*, *c*, …; the elements of $\mathcal{P}$ are called points and denoted by *A*, *B*, *C*, …; two lines are *perpendicular* (*orthogonal*), $a \perp b$, iff $a \mid b$. A point *A* and a line *b* are *incident* if $A \mid b$. $A = a$ means that *they are polar to each other*, such point and line not necessarily exist (their existence characterizes the *elliptic* (*spherical*) *plane*. $d \in \mathcal{S}$ is *the fourth reflection line* to *a*, *b*, *c*, if $d = abc$ (Fig. 3). We assume that the next axioms hold (please formulate them by usual sentences):

A.1 To *A*, *B* there exists *c* with $A, B \mid c$ (i.e. $A \mid c$ and $B \mid c$, similar shortening will be used later on).

A.2 $A, B \mid$ c, d imply $A = B$ or $c = d$.

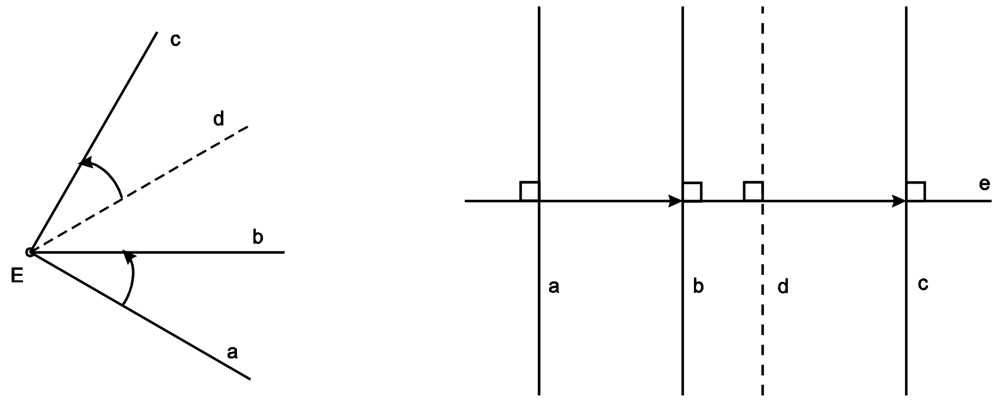


Fig. 3. For axioms A.3 and A.4 of three reflections

A.3 $a, b, c \mid E$ imply $abc \in \mathcal{S}$.

A.4 $a, b, c \mid e$ imply $abc \in \mathcal{S}$.

A.D (*On "rectangular triangle"*) There exist *g*, *h*, *j* such that $g \mid h$ and neither $j \mid g$ nor $j \mid h$ nor $j \mid gh$.

Now we only mention from [1] the

Main Theorem. Each group plane $\mathcal{M}$ can be embedded into a projective metric plane $\mathcal{P}^2$(**K**, $\mathbf{V}^3$, $\mathbf{V}_3$, ~, $\langle,\rangle$), as above (in Sect.1), with some additional conditions on scalar product $\langle,\rangle$, indicated above. □

Here line reflection and point one, both will be harmonic homology σ_A^a, i.e. $A = a$. Such a reflection does not exist for $A \mid a$, then we speak on boundary point and boundary line, whose existence characterizes e.g. the hyperbolic geometry, then we also say that polar and pole are incident.

As we see, axioms A.3 and A.4 require the so-called three reflection theorems in two specific cases (Fig. 3). That case, where two lines a, b neither have common point E nor have common perpendicular e, will be a nice theorem for the so called *parallel lines*, constructing also a *third parallel lines* b to a and c through any given point (say) X, Fig. 4. This leads us to introducing line pencil as ideal point and other concepts of projective embedding in subsections 2.1 and 2.2, as our honour to J. Hjelmslev [3].

2. ON PROJECTIVE REFLECTION, THE GENERALIZED THEOREM OF THREE REFLECTIONS

The reflection σ_U^u on the above projective plane $\mathbf{P}^2(\mathbf{K}, \mathbf{V}^3, \boldsymbol{V}_3, \sim)$ with *Char* $\mathbf{K} \neq 2$ can be given by equations

$$\sigma_U^u\colon V3 \to \mathbf{V}^3,\ \mathbf{X} \to \mathbf{X}^{\sigma_U^u} = \mathbf{Y} = \mathbf{X} - \frac{2(\mathbf{X}\boldsymbol{u})}{(\mathbf{U}\boldsymbol{u})}\mathbf{U} \text{ and } \boldsymbol{V}_3 \to \boldsymbol{V}_3,\ \boldsymbol{x} \to \boldsymbol{x}^{\sigma_U^u} = \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{u}\frac{(\mathbf{U}\boldsymbol{x})}{(\mathbf{U}\boldsymbol{u})}2 \qquad (2.1)$$

for points and lines, respectively.

Definition 2.1 (of rotation and point reflection). Reflections σ_A^a and σ_B^b define a projective rotation

$$\mathbf{r} = \sigma_A^a\sigma_B^b\colon\ \mathbf{X} \to \mathbf{Y} = \mathbf{X} - 2\frac{(\mathbf{X}\boldsymbol{a})}{(\mathbf{A}\boldsymbol{a})}\mathbf{A} - 2\frac{(\mathbf{X}\boldsymbol{b})}{(\mathbf{B}\boldsymbol{b})}\mathbf{B} + 4\frac{(\mathbf{X}\boldsymbol{a})(\mathbf{A}\boldsymbol{b})}{(\mathbf{A}\boldsymbol{a})(\mathbf{B}\boldsymbol{b})}\mathbf{B} \quad \text{and}$$
$$\boldsymbol{x} \to \boldsymbol{y} = \mathbf{x} - \boldsymbol{a}\frac{(\mathbf{A}\boldsymbol{x})}{(\mathbf{A}\boldsymbol{a})}2 - \boldsymbol{b}\frac{(\mathbf{B}\boldsymbol{x})}{(\mathbf{B}\boldsymbol{b})}2 + \boldsymbol{b}\frac{(\mathbf{B}\boldsymbol{a})(\mathbf{A}\boldsymbol{x})}{(\mathbf{B}\boldsymbol{b})(\mathbf{A}\boldsymbol{a})}4 \qquad (2.2)$$

again for points and lines, respectively, if $\boldsymbol{a} \neq \boldsymbol{b}$ and $\mathbf{A} \neq \mathbf{B}$. Then $a \cap b =: ab =: R$ is called the fixed point of rotation (not a centre, in general), $A \cup B =: AB =: r$ is called the fixed line of rotation (not an axis, in general).

If $B \in a$ and $A \in b$, then $\mathbf{r} = \boldsymbol{R}$ is called point reflection $\sigma_R^r = \sigma_A^a\sigma_B^b = \sigma_B^b\sigma_A^a$ (see also Fig. 1 in other roles, W will be the centre of that) as *involutive rotation*. □

Definition 2.2 (of *translation*). Translation τ_{AB}^t is the collineation of $\mathbf{P}^2(\mathbf{K}, \mathbf{V}^3, \boldsymbol{V}_3, \sim)$ with incident centre T and axis t, mapping a point A into a point $B = A^\tau$ (Fig. 2 shows the dual case) so that $T = t \cap AB$. □

With simple verification we obtain

Theorem 2.3. Two reflections with the same axis t = a = b compose the translation $\tau_{AC}^t = \sigma_A^t\sigma_B^t$ where

$$C = A^{\sigma_B^t},\ \textit{and}\ \sigma_A^t\ \sigma_B^t\colon\ \mathbf{X} \to \mathbf{Y} = \mathbf{X} - \frac{2(\mathbf{X}\boldsymbol{t})}{\mathbf{A}\boldsymbol{t}}\mathbf{A} + \frac{2(\mathbf{X}\boldsymbol{t})}{(\mathbf{B}\boldsymbol{t})}\mathbf{B}\ \textit{for points and}$$
$$\boldsymbol{x} \to \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{t}\left(\frac{2(\mathbf{A}\boldsymbol{x})}{(\mathbf{A}\boldsymbol{t})} - \frac{2(\mathbf{B}\mathbf{x})}{(\mathbf{B}\boldsymbol{t})}\right)\ \textit{for lines.}$$
$$\textit{The centre is } \mathbf{T} = \frac{\mathbf{A}}{(\mathbf{A}\boldsymbol{t})} - \frac{\mathbf{B}}{(\mathbf{B}\boldsymbol{t})}. \ \square$$

The dual definition and theorem of the previous ones, respectively, can also be easily formulated. We can check our following main statements with similarly straightforward but lengthy computations:

Theorem 2.4 (of three reflections, or of two rotations or of two translations). The composition (product) of any three reflections is a fourth reflection, i.e. $\sigma_A^a\sigma_B^b\sigma_C^c = \sigma_D^d$; or we obtain equal rotations or equal translations $\sigma_A^a\sigma_B^b = \sigma_D^d\sigma_C^c$ with appropriate axis d and centre D, if and only if:

a) *In case $a \neq b$ an $A \neq B$; then a, b, c, d are incident to a point R, and A, B, C, D are incident to a line r, the polar to R. This can be expressed with vectors* **A**, **B**, **C**, **D** *and the corresponding polar forms* ***a***, ***b***, ***c***, ***d*** *so that, for the above scalar product, defined in (1.1), holds e.g. for forms*
$\boldsymbol{d} = \boldsymbol{a}\langle \boldsymbol{b}, \boldsymbol{c}\rangle - \boldsymbol{b}\langle \boldsymbol{a}, \boldsymbol{c}\rangle + \boldsymbol{c}\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ *with linearly dependent forms* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} = \boldsymbol{a}\alpha + \boldsymbol{b}\beta$;
$\langle \boldsymbol{d}, \boldsymbol{c}\rangle^2/\langle \boldsymbol{d}, \boldsymbol{d}\rangle\langle \boldsymbol{c}, \boldsymbol{c}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle^2/\langle \boldsymbol{a}, \boldsymbol{a}\rangle\langle \boldsymbol{b}, \boldsymbol{b}\rangle$.

b) *In case $a = b = t$ but $A \neq B$ we get equal translations*
$\sigma_A^t\ \sigma_B^t = \sigma_D^t\ \sigma_C^t$, *i.e.* $c = d = t$ *and* $\tau_{AB}^t = \tau_{DC}^t$;

c) *In dual case $a \neq b$ but $A = B = T$ we get equal dual translations*
$\sigma_T^a\ \sigma_T^b = \sigma_T^d\ \sigma_T^c$, *i.e.* $C = D = T$ *and* $\boldsymbol{\tau}_{\boldsymbol{T}}^{\boldsymbol{ab}} = \boldsymbol{\tau}_{\boldsymbol{T}}^{\boldsymbol{dc}}$. □

2.1. On halfrotations and the projective embedding. Analytic form of halfrotation

As we have already mentioned to Axioms 3-4, there is a nice consequence of them.

Theorem 2.5 (of perpendiculars, Lotensatz of Hjelmslev, Fig. 4) *To any lines $a \neq c$ and any point X ($\neq a \cap c$) there is a unique line b such that $b \mid X$ and $abc = d$* (in the sense of Theorem 2.4).
Proof (Sketch). There are a’,c’,d’ | X such that $a'a = A = \bar{a}x$; $c'c = C = x\bar{c}$; $x \mid \bar{a}, \bar{c}, d'$; $Ad'C = \bar{a}xd'x\bar{c} = \bar{a}d'\bar{c} = d$; $a'd'c' = b$, by Axioms 3-4 and their completions.

Thus we get $abc = a(a'd'c')c = Ad'C = d$, as desired. □

Then we can define and introduce such line pencil b,d $\in \mathcal{I}$(a,c) as an *ideal point* $\mathcal{I}$(a,b,c,d). We also see proper point $X = \mathcal{I}$(a’,b,d’,c’), orthogonal line pencil $\mathcal{I}$(x) $\mathcal{I}(\bar{a},\bar{c}$,d) as ideal points in Fig. 4. Moreover, we can define an important transform, the so called halfrotation (Halbdrehung of Hjelmslev for defining an embedding projective plane structure with ideal points and ideal lines. Furthermore, orthogonality of lines will be defined with a symmetric polarity as line (polar) $\rightarrow$ point (pole) mapping. To this will be important the following (surprisingly) simple construction and its symbolic description

Theorem 2.6 (of foursides, Fig. 5) If a’ ∤ b’, $a'b' = d'c'$, $A = aa'$, $C = c'c$, $b = (A, b')$, $d = (C, d')$, then $c \mid b \Leftrightarrow a \mid d$.
Proof follows from the fourside constructions by equalities $X = Ab'c = aa'b'c = ad'c'c = ad'C$. □

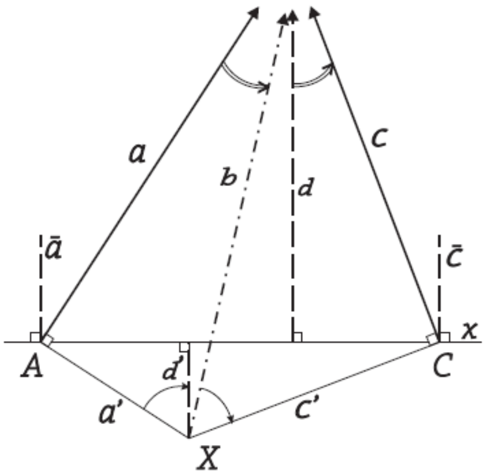


Fig. 4 To the theorem of perpendiculars (Lotensatz)

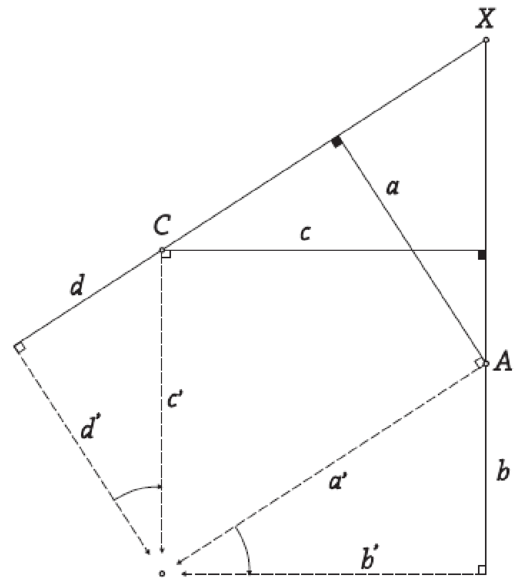


Fig.5 To the foursides theorem

Definition 2.3 (of halfrotation Fig. 6) *Let two lines u and v* ($u \nmid v$, i.e. they are different and not perpendicular through an (ideal) point *O*). *The halfrotation* $[uv] = {}^*$ *as a point mapping* $X \to X^*$*, and a line mapping* $x \to x^*$ *will be given* (to the generalized rotation *uv*).

To x, not incident to O, we order x^* *the fixed line of the mapping* $xuv = xx'x'^* = Xx'^*$. *To x, through O, let* $x^* = xuv$.

To $X = xx'$ *above, let* $X^* = x^* x'^*$. (In Fig. 6 we only indicate that X^* is the midpoint of *X and* X^{uv} . □

The most important property of halfrotation that it preserves incidence of lines to ideal points.

Theorem 2.7 Let $[uv] = {}^*$ be a halfrotation about O. Then $abc \in \mathcal{S} \Leftrightarrow a^* b^* c^* \in \mathcal{S}$. The image of proper ideal point is also proper one. □

Fig. 7 shows only a specific case connecting it with the Theorem 2.5 of perpendiculars

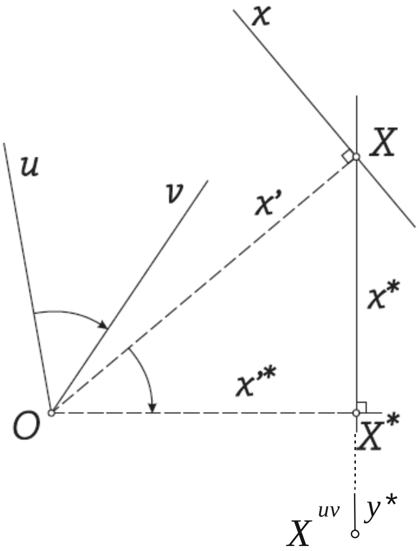
$y, a', a \mid A \Rightarrow y^*, a'^*, a^* \mid A^*$.

This situation is similar as at

Theorem 2.8 Halfrotations about fixed O commute (Fig. 8). □

Then we can close *projective embedding* by defining *ideal line*, as a set of ideal points whose images at certain (finite) composition of halfrotations will be ideal points with common proper line (See [1] for more details).

Again a deep theorem that our embedding will be a Pappian projective plane. Then it is Desargues by the theorem of Hessenberg, then a commutative coordinate field K can also be introduced.



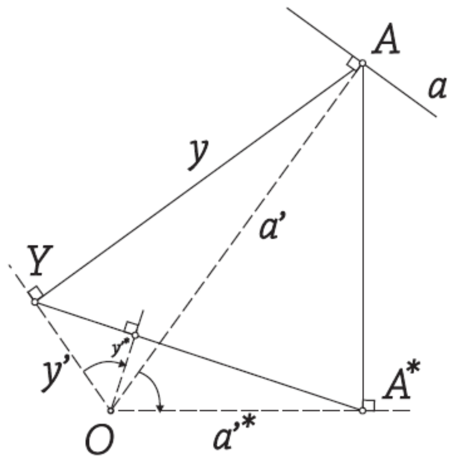


Fig. 6. To definition of the halfrotation $[uv] = {}^{*}$: $X \to X^{*}$, $x \to x^{*}$
Fig. 7. Connection with the theorem of perpendiculars (Lotensatz, see Fig.4)

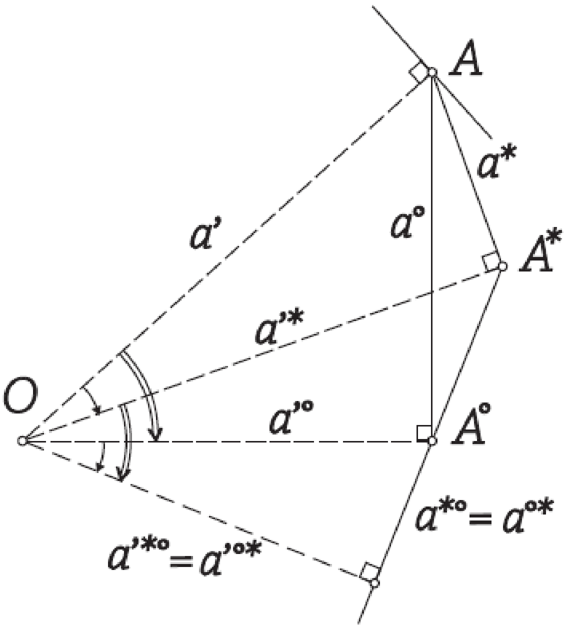


Fig. 8 Halfrotations commute by Lotensatz 2.5

The promised projective polarity is based on

Theorem 2.9 (on opposite halfrotations Fig. 9) Let $[uv] = {}^{*}$ and $[vu] = {}_{*}$ be opposite halfrotations, and let. $\iota(a)$ and so $\iota(a^{*})$ proper ideal lines. Let $I(a)$ and $I(a^{*})$ be the corresponding perpendicular pencils as ideal points. Then $I(a^{*})_{*} = I(a)$.
Proof without words: $a^{*} \mid c \Leftrightarrow a \mid c_{*}$, as desired. □

As we have seen, halfrotation does not preserve orthogonality of lines, but opposite halfrotations make the important connections.

Definition 2.4 (of polarity) An ideal line x and an ideal point X are called polar pole pair if there is a halfrotation [uv] = * about O and a proper ideal line ι(a) so that $x^* = ι(a)$ and $I(a)_* =$ X. ι(a) and I(a) are called primitive polar-pole pair. Moreover, let o = ι(O) and O = I(O) be also polar pole pair. □

Fig. 10 shows also a construction (besides other ones), how to determine the polar (ideal) line $\iota(U)$ of a proper point U? Namely, $\iota(U) = \{\mathcal{I}(y) : y \mid U\}$.

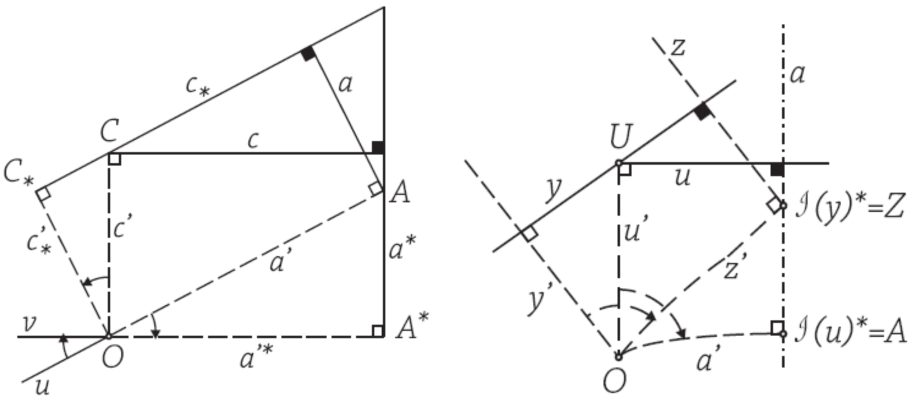


Fig. 9 Polarity and its connection with opposite halfrotations * = [uv] and $_*$ = [vu] (see also Fig. 5)
Fig.10 Construction of the polar line $\iota(U)$ of U and other ones (Theorem. 2.6)

Halfrotations make other nice connections as well.

But what is its analytic description in the projective plane $\mathcal{P}^2$(K, V^3, V$_3$, ~, ⟨,⟩) that it has derived? That will be a new(?) initiative of the author also for other (seemingly open?) problems.

Theorem 2.10 (Analytic formulas of halfrotation in projective metric plane for points and dually for lines) *Let* [ab] = *: $X \to X^*$, $y \to y^*$ *, given by projective reflections* $a \to \sigma_A^a$ *and* $b \to \sigma_B^b$

$$\tfrac{1}{2}(\mathbf{1}+\sigma_A^a\sigma_B^b):\ \mathbf{X} \to \mathbf{X}^* = \mathbf{X} - \frac{(\mathbf{X}\boldsymbol{a})}{(\mathbf{A}\boldsymbol{a})}\mathbf{A} - \frac{(\mathbf{X}\boldsymbol{b})}{(\mathbf{B}\boldsymbol{b})}\mathbf{B} + 2\frac{(\mathbf{X}\boldsymbol{a})(\mathbf{A}\boldsymbol{b})}{(\mathbf{A}\boldsymbol{a})(\mathbf{B}\boldsymbol{b})}\mathbf{B}\ \text{ and}$$

$$\boldsymbol{y} \to \boldsymbol{y}^* = \mathbf{y} - \boldsymbol{a}\frac{(\mathbf{A}\boldsymbol{y})}{(\mathbf{A}\boldsymbol{a})} - \boldsymbol{b}\frac{(\mathbf{B}\boldsymbol{y})}{(\mathbf{B}\boldsymbol{b})} + \boldsymbol{b}\frac{(\mathbf{B}\boldsymbol{a})(\mathbf{A}\boldsymbol{y})}{(\mathbf{B}\boldsymbol{b})(\mathbf{A}\boldsymbol{a})}2$$

Proof is easy consequence of formulas (2.2)

2.2. The main (open!?) problem: Models of Bachmann's metric plane

How to characterize the field K and the projective metric plane $\mathcal{P}^2$(K, V^3, V_3, ~, ⟨,⟩), so that we obtain a model of a metric plane $M(G,S)$ in 1.2 with the given projective embedding?

We would like to turn back to this problem. The interested Readers are also kindly invited. See also [1], editions (1969 and 1973).

3. CONFIGURATION THEOREMS FOR CYCLES

We have already introduced the concept of *cycle* as reflection images of a starting point A_0 in all lines of the pencil through the centre A in (1.3), Fig. 11. It is important to mention that – in the extended sense of the three reflection theorem (in Sect. 2) – outer and boundary centre (e.g. for hypercycle and horocycle), outer and boundary line a_i through it, moreover outer and boundary point A_i for a cycle point come into consideration in the moveable equivalence class of the starting A_0. Moreover, each (no-boundary) point of $\mathbf{H}^2$ and $\mathbf{S}^2$ *has to be doubled, because the inversion makes difference in the images for a point and its duplicate*. *The boundary points of* $\mathbf{H}^2$, each is incident to its boundary polar, *will belong to every moveable class*, these would cause "easy but lengthy difficulties" in the comprehensive discussion that we leave out here for the sake of brevity. Lines can also be considered as *not ordinary* cycles in the above sense. *We mention the basic tools of our inversion concept*

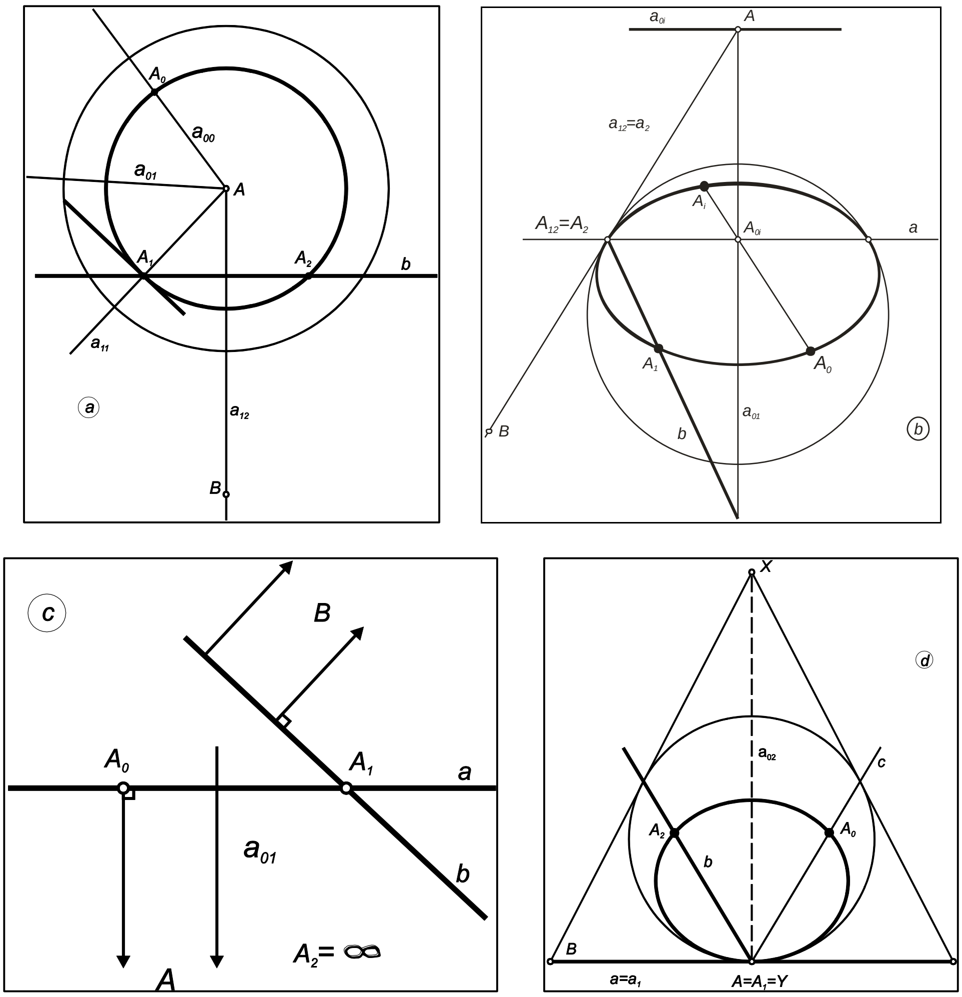


Fig. 11. Cycles $\mathcal{C}(A, A_0)$ and its intersection with line b

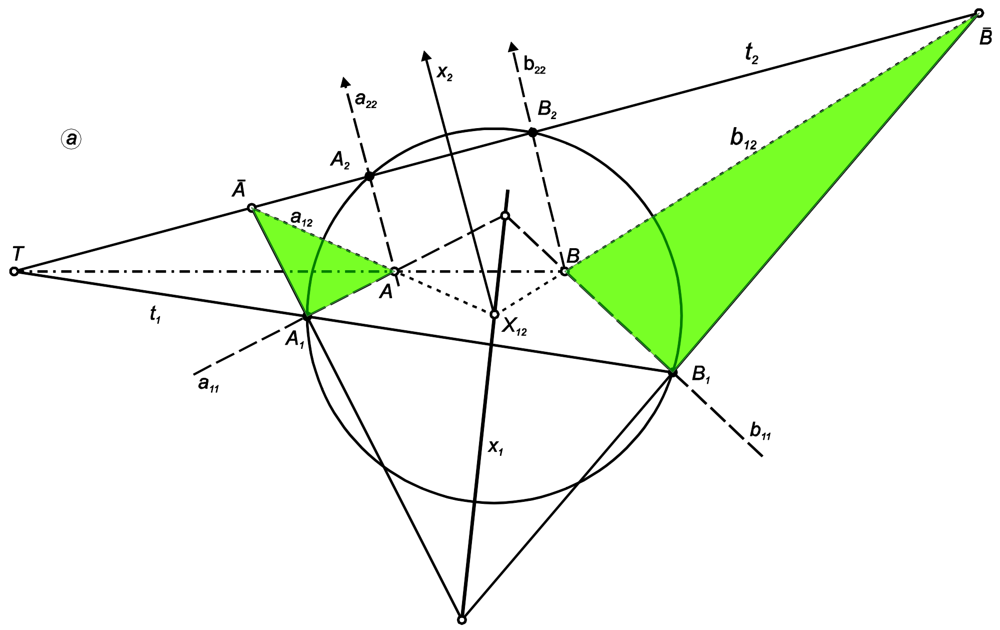


Fig. 12. The Chord theorem as a consequence of Desargues' one

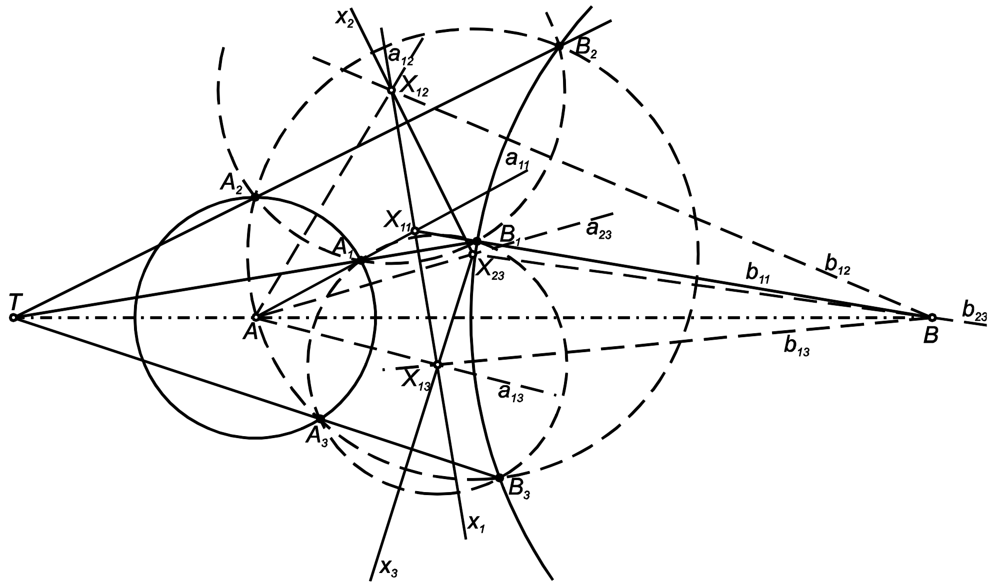


Fig. 13. The Pencil theorem for the existence of inversion with centre T (a fixed circle does *not necessarily exist)*

Theorem 3.1 (Chord theorem, Fig. 12). Assume that $A_1 \neq A_2$, $B_1 \neq B_2$ hold for the points of an ordinary cycle $\mathcal{C}(X_{12}, A_1)$ for the line x_1, a_{12}, x_2, b_{12} through the centre X_{12} (in Definition 1.2, (1.4)) and for the line a_{11}, b_{11}, a_{22}, b_{22}, through A_1, B_1, A_2, B_2, respectively;

a) Let a_{11}, a_{12}, a_{22} have the point A; b_{11}, b_{12}, b_{22} have the point B; in common for the cycles
$\mathcal{C}(A, A_1)$ and $\mathcal{C}(B, B_1)$, respectively;

b) Let line $t_1 := (A_1, x_1, B_1)$ be incident to A_1, B_1, and orthogonal to x_1, and let $t_2 := (A_2, x_2, B_2)$ similarly be defined.

Then A, B, T lie in a line.

Proof. The picture (Fig. 12) shows, without any word, the consequence by the Theorem of Desargues on perspective triangles. □

Theorem 3.2 (Pencil theorem, Fig. 13). Consider three lines in the extended configuration of Fig. 12, $t_1 := (A_1, x_1, B_1)$, $t_2 := (A_2, x_2, B_2)$, $t_3 := (A_3, x_3, B_3)$. These lines have then a point T in common (Fig. 13). □

Theorem 3.3 (Theorem of Miquel). Consider six point-quadruple (A_1, A_2, A_3, A_4), (A_1, B_1, B_2, A_2), (A_2, B_2, B_3, A_3), (A_3, B_3, B_4, A_4), (A_4, B_4, B_1, A_1), (B_1, B_2, B_3, B_4) in an extended equivalence class of moveable points. If the first five quadruple lie in a cycle, respectively, then this holds also for the sixth quadruple.

Proof is typical consequence by many (six!) applications of the theorem of three reflections (see only (1.4), we do not give the well-known figure here). □

4. INVERSION IN A MOVEABLE CLASS OF A PROJECTIVE METRIC PLANE

Our preliminary Definition 1.2 in the introduction is living now by the above Pencil theorem (Th. 3.2, Fig. 13). An analytic description will support this, providing the higher dimensional extension, too. First, think of $\mathbf{S}^2$ and $\mathbf{H}^2$ embedded into $\mathcal{A}\mathbf{V}^3$ (and $\mathbf{P}^3$) as affine interpretation, with doubled points $\mathbf{X}$ and $-\mathbf{X}$ symmetric in the affine origin $O^{\sim}$ if $\langle \mathbf{X}, \mathbf{X}\rangle \neq 0$. The boundary points $\mathcal{H}$ of $\mathbf{H}^2$ with equation $\langle \mathbf{X}, \mathbf{X}\rangle = 0$ will be sent to infinity with a new coordinate $X^{-1} = 0$ in a new projective embedding into $\mathbf{P}^3 = \mathbf{P}^{n+1}$. The other points get coordinates $X^{-1} = 1$, (say) first.

The equation of cycle (*a type of hypersphere in higher dimensions*) *will be* – on the base of (1.3):

$$\langle \mathbf{A}, \mathbf{A}_i - \mathbf{A}_0\rangle = 0 \quad \textit{with fixed} \quad \langle \mathbf{A}_0, \mathbf{A}_0\rangle = \langle \mathbf{A}_i, \mathbf{A}_i \rangle = -d \in \mathbf{K} \setminus 0, \tag{4.1}$$

for $\mathbf{A}_i$ as variable cycle point, $\mathbf{A}$ is the given centre, $\mathbf{A}_0$ is a given cycle point. That means, cycle is a plane (or hyperplane) intersection of a quadric. □

4.1. Spherical and hyperbolic inversion embedded into $\mathbf{P}^3 = \mathbf{P}^{n+1}$

Let us a new basis vector $\mathbf{E}_{-1}$ be introduced for a new vector space (still now we used $\mathbf{V}^3 = \mathbf{V}^{n+1}$ with basis $\mathbf{E}_0$, , $\mathbf{E}_1, \dots, \mathbf{E}_n$). That means, we introduce $\mathbf{P}^3 = \mathbf{P}^{n+1}$ with $\mathbf{V}^{n+2} = \mathbf{E}_{-1} \oplus \mathbf{V}^{n+1}$ and scalar product $\langle \mathbf{E}_{-1} \oplus \mathbf{X}, \mathbf{E}_{-1} \oplus \mathbf{Y} \rangle = d + \langle \mathbf{X},\mathbf{Y}\rangle$. That means, our cycle or hypersphere remains a hyperplane intersection of a new quadric (now in zero level), in one dimension higher, as in classical case

The origin $O \to O^{\sim}$ of $\mathcal{A}\mathbf{V}^3 = \mathcal{A}\mathbf{V}^{n+1}$ will be $\mathbf{E}_{-1} + \mathbf{0} =: \mathbf{O}^{\sim}$ whose polar hyperplane, related to the new scalar product, will be $\boldsymbol{e}^{-1} + \boldsymbol{o} =: \boldsymbol{o}^{\sim}$. We do not detail the further technical circumstances of the embedding into the n+1-dimensional projective metric space $\mathbf{P}^3 = \mathbf{P}^{n+1}$.

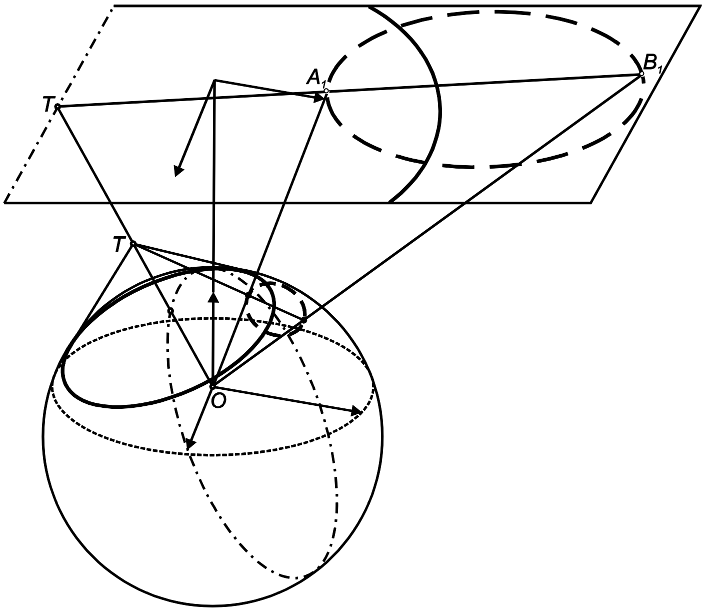


Fig. 14. Spherical inversion from higher dimensional projection

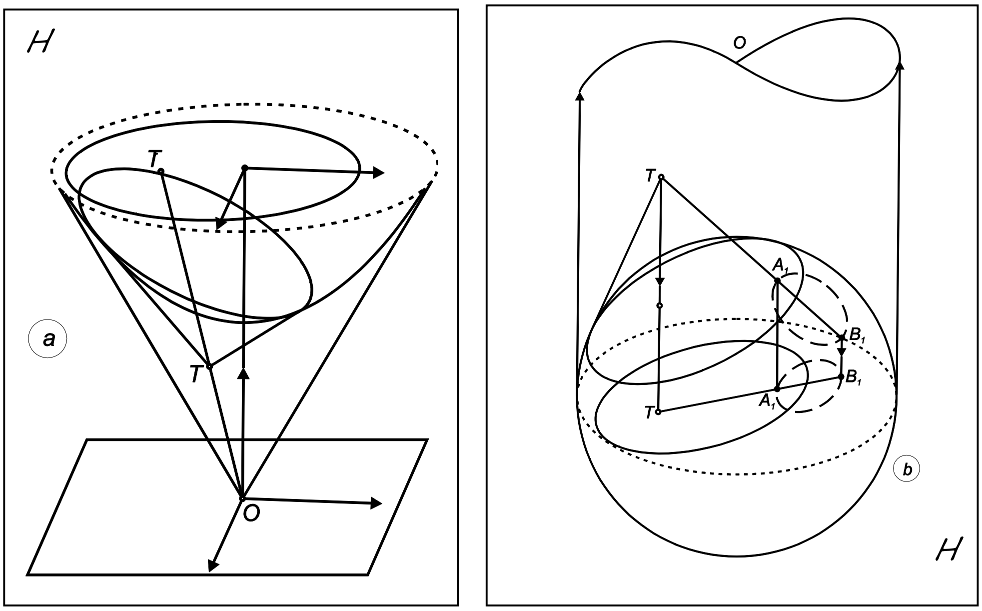


Fig. 15. Hyperbolic inversion from higher dimensional projection; the absolute H lies in different positions

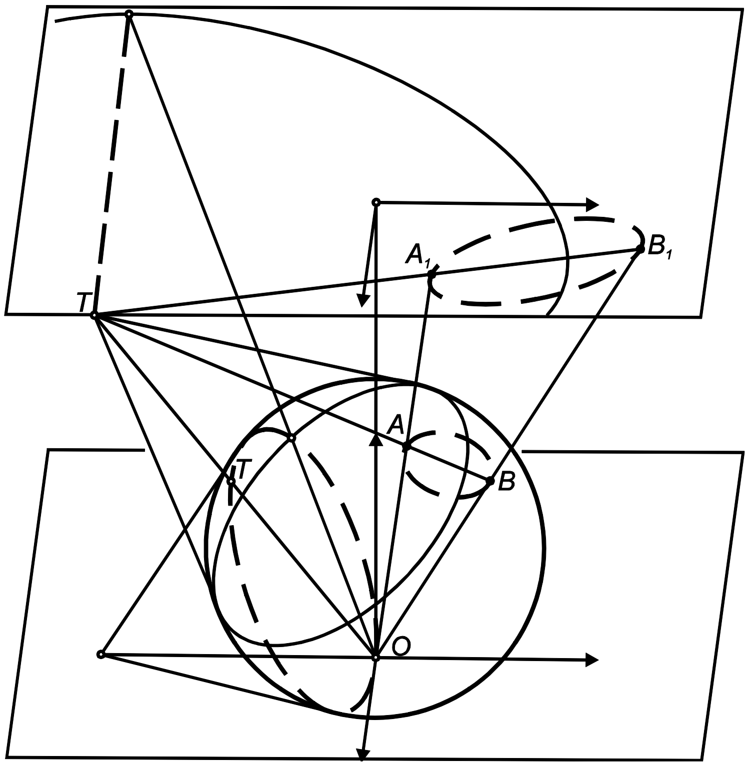


Fig. 16. Euclidean inversion by stereographic projection from O $\to$ O$^{\sim}$

Now we are ready to formulate our main theorem for spherical or hyperbolic inversion.

Theorem 4.1 (Inversions in $\mathbf{S}^2 = \mathbf{S}^n$ and $\mathbf{H}^2 = \mathbf{H}^n$). The inversion α(T, A_1, x_1, B_1), defined above on a (generalized) doubled point class $\mathcal{D}$, can be presented as a projective reflection in the projective embedding $\mathbf{P}^3 = \mathbf{P}^{n+1}$, where the absolute quadric $\mathcal{D}^{\sim}$, describing $\mathcal{D}$, is mapped onto itself (Fig. 14-16).

Proof (sketch). We follow Definition 1.2 at (1.4). Let the inversion centre $T(\mathbf{T})$ in $\mathbf{P}^2 = \mathbf{P}^n$ and $A_1(\mathbf{A}_1)$ with $\langle \mathbf{A}_1, \mathbf{A}_1\rangle = -d$ be fixed, and consider the quadric $\mathcal{D}^{\sim}$ defined by A_1 in $\mathcal{A}\mathbf{V}^{n+1}$ and so in $\mathbf{P}^3 = \mathbf{P}^{n+1}$. The above line x_1 will be given with its pole X_1

$$\mathbf{X}_1 \sim \mathbf{T} - T^{-1}\mathbf{A}_1,\ X_1 \neq T,\ \langle \mathbf{X}_1, \mathbf{X}_1\rangle \neq 0,\ \langle \mathbf{T}, \mathbf{T}\rangle \neq T^{-1}T^{-1}\langle \mathbf{A}_1, \mathbf{A}_1\rangle = -\,T^{-1}T^{-1}d \qquad (4.2)$$

and let this essential coefficient T^{-1} be fixed. The image $B_1(\mathbf{B}_1)$ of $A_1(\mathbf{A}_1)$ is then defined by (1.4)

$$\mathbf{B}_1 = \mathbf{A}_1 - 2\langle \mathbf{A}_1,\mathbf{X}_1\rangle\, \mathbf{X}_1 \big/ \langle \mathbf{X}_1,\mathbf{X}_1\rangle\ = \qquad (4.3)$$

$$[(\langle \mathbf{T}, \mathbf{T}\rangle + T^{-1}T^{-1}d)\, \mathbf{A}_1 - 2(\langle \mathbf{T}, \mathbf{A}_1\rangle + T^{-1}d)\, \mathbf{T}] \,/\, [(\langle \mathbf{T}, \mathbf{T}\rangle + T^{-1}T^{-1}d) - 2(\langle \mathbf{T}, \mathbf{A}_1\rangle + T^{-1}d)\, T^{-1}] =$$

$$\{\mathbf{A}_1 - 2[(\langle \mathbf{T}, \mathbf{A}_1\rangle + T^{-1}d)\, \mathbf{T}/(\langle \mathbf{T}, \mathbf{T}\rangle + T^{-1}T^{-1}d)]\} \,\big/\, \{1 - 2[(\langle \mathbf{T}, \mathbf{A}_1\rangle + T^{-1}d)\, \mathbf{T}/(\langle \mathbf{T}, \mathbf{T}\rangle + T^{-1}T^{-}d)]\, T^{-1}\}.$$

If the point $T^{\sim}$ is described by $\mathbf{T}^{\sim} = T^{-1}\mathbf{E}_{-1} + \mathbf{T}$ in $\mathbf{P}^3 = \mathbf{P}^{n+1}$, moreover its polar hyperplane $t^{\sim}$ is then $\boldsymbol{t}^{\sim} = \boldsymbol{e}^{-1}d\, T^{-1} + \boldsymbol{t}$, then the above mapping (4.3) can be written in $\mathbf{P}^3 = \mathbf{P}^{n+1}$ as a reflection formula (Fig. 14-15):

$$\mathbf{B}^{\sim} = \mathbf{A}^{\sim} - 2\langle \mathbf{A}^{\sim}, \mathbf{T}^{\sim}\rangle \mathbf{T}^{\sim} / \langle \mathbf{T}^{\sim}, \mathbf{T}^{\sim}\rangle. \square \tag{4.4}$$

Remark 4.2. As we have seen above, the quadratic classes of the basic field $\mathbf{K}$ partition the points of the projective extension of $\mathbf{S}^2 = \mathbf{S}^n$ and $\mathbf{H}^2 = \mathbf{H}^n$, respectively, into equivalence classes of doubled moveable points. An inversion acts exclusively on one class as on a quadric.

This is an essential generalization also for classical Poincaré model of $\mathbf{H}^n$, related to the case $\mathbf{K} = \mathbf{R}$ (real numbers), where we have only two quadratic classes. Complex number field $\mathbf{K} = \mathbf{C}$ is another subject, the finite fields are also prospective [4]. □

Euclidean and Minkowski planes, respectively, where the poles of lines fall onto an ideal line, forming there an orthogonality involution, mean degenerate cases. Overview of inversions on the first one is well-known as it briefly follows below. *The second one, i.e.* ***inversions on $\mathbf{M}^2$ or on $\mathbf{M}^n$, seems to be a timely research topic (!?!)***.

4.2. Euclidean type inversion by stereographic projection, embedded into $\mathbf{P}^3 = \mathbf{P}^{n+1}$

For this more familiar topic, in the Euclidean case $\mathbf{E}^2 = \mathbf{E}^n$, we promptly give the *projective-conform interpretation*, but in more general form (so that you could immediately extend it to $\mathbf{M}^2 = \mathbf{M}^n$ as well, see also Fig. 16).

The projective metric plane $\mathbf{P}^2 = \mathbf{P}^n$ has already been given above by $\mathbf{V}^3 = \mathbf{V}^{n+1}$ with a basis $\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_2 = \mathbf{E}_n$ (and its dual basis was $\boldsymbol{e}^0, \boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$ with $(\mathbf{E}_i\boldsymbol{e}^k) = \delta_i^k$ the Kronecker symbol). The previous bilinear form (scalar product) was $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_0 \cdot 0 \cdot y_0 + x_\alpha \pi^{\alpha\beta} y_\beta$ (Greek indices run from 1 to n with Einstein-Schouten sum convention). That means, $\boldsymbol{e}^0$ (line by equation $X^0 = 0$) describes the ideal line (hyperplane) with an elliptic involution on it, in case $n = 2$ (elliptic polarity in general $n > 2$). We start in the extended dual $\mathbf{K}$ vector space $\boldsymbol{V}_4 = \boldsymbol{V}_{n+2} = \boldsymbol{e}^{-1} \oplus \boldsymbol{V}_{n+1}$ with an extended scalar product

$$\langle \boldsymbol{x}^{\sim}, \boldsymbol{y}^{\sim}\rangle := -(1/ccd)\, x_{-1}y_{-1} - (1/cd)\, x_{-1}y_0 - (1/cd)\, x_0 y_{-1} + x_\alpha \pi^{\alpha\beta} y_\beta \quad \text{with fixed } 0 \neq c, d \in \mathbf{K}. \tag{4.5}$$

This will uniquely be related to that of the vector space $\mathbf{V}^{n+2} = \mathbf{E}_{-1} \oplus \mathbf{V}^{n+1}$

$$\langle \mathbf{A}^{\sim}, \mathbf{B}^{\sim}\rangle := -cdA^{-1}B^0 - cd\,A^0B^{-1} + d\,A^0B^0 + A^\beta \Pi_{\beta\gamma} B^\gamma, \text{ with previous fixed c, d, and } \Pi_{\alpha\beta}\cdot\pi^{\beta\gamma} = \delta_\alpha^\gamma. \tag{4.6}$$

We look at (4.5-6) that the hyperplane $^{\sim}\boldsymbol{e}^0$ of $\mathbf{P}^3 = \mathbf{P}^{n+1}$ that corresponds to the ideal line (hyperplane) of $\mathbf{P}^2 = \mathbf{P}^n$ touches in the origin $\mathbf{O}^{\sim}(\mathbf{E}_{-1})$ the quadric

$$\mathcal{D}^{\sim} = \{A^{\sim}(\mathbf{A}^{\sim}) \in \mathcal{A}\mathbf{V}^3 = \mathbf{P}^3 = \mathbf{P}^{n+1} \mid \langle \mathbf{A}^{\sim}, \mathbf{A}^{\sim}\rangle = 0\}. \tag{4.7}$$

Lines (hyperplanes) $x(\boldsymbol{x})$ and $y(\boldsymbol{y})$ in $\mathbf{P}^2 = \mathbf{P}^n$ are orthogonal, i.e. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$, if $\langle \boldsymbol{x}^{\sim}, \boldsymbol{y}^{\sim}\rangle = 0$ holds for the according planes (hyperplanes) $x^{\sim}(\boldsymbol{x}^{\sim} := \boldsymbol{e}^{-1}\cdot 0 + \boldsymbol{x})$ and $y^{\sim}(\boldsymbol{y}^{\sim} := \boldsymbol{e}^{-1}\cdot 0 + \boldsymbol{y})$ both are incident to the origin $O^{\sim}(\mathbf{E}_{-1})$ in $\mathbf{P}^3 = \mathbf{P}^{n+1}$. The points of $\mathbf{P}^2 = \mathbf{P}^n$ can stereographically be projected onto the above quadric as follows. The points of $\boldsymbol{e}^0$ will be unified into a symbolic infinity point ∞. This will be ordered to the origin $O^{\sim}(\mathbf{E}_{-1})$ of $\mathbf{P}^3 = \mathbf{P}^{n+1}$. The projection formula will be

$$\mathbf{P}^n \setminus e^0 \to \mathcal{D}^{\sim} \setminus O^{\sim}: \; A\big(\mathbf{E}_0 + A^\alpha \mathbf{E}_\alpha\big) \to A^{\sim}\big((1/2cd)(d + A^\alpha \Pi_{\alpha\beta} B^\beta)\cdot \mathbf{E}_{-1} + \mathbf{E}_0 + A^\alpha \mathbf{E}_\alpha\big). \tag{4.8}$$

The reflection of the form

$$A^{\sim}(\mathbf{A}^{\sim}) \to B^{\sim}(\mathbf{B}^{\sim}) : \mathbf{B}^{\sim} = \mathbf{A}^{\sim} - 2\langle \mathbf{A}^{\sim}, \mathbf{X}^{\sim}\rangle\, \mathbf{X}^{\sim}/\langle \mathbf{X}^{\sim}, \mathbf{X}^{\sim}\rangle, \tag{4.9}$$

with axis hyperplane $x^{\sim}(\boldsymbol{x}^{\sim} := \boldsymbol{e}^{-1}\cdot 0 + \boldsymbol{x})$, through the origin $O^{\sim}(\mathbf{E}_{-1})$, and with its pole

$$X^{\sim}\,(\,\mathbf{X}^{\sim} = -\,(1/cd)\; x_0\cdot\mathbf{E}_{-1} + x_\alpha\pi^{\alpha\beta}\cdot\mathbf{E}_\beta),$$

maps the quadric $\mathcal{D}^{\sim}$ (in $\mathbf{P}^3 = \mathbf{P}^{n+1}$) onto itself. As a summary we formulate the following

Theorem 4.3 (On Euclidean inversion and stereographic projection). A cycle (hypersphere) $\mathcal{C}(\mathrm{A}, \mathrm{A}_0)$ of the Euclidean projective metric plane $\mathbf{P}^2 = \mathbf{P}^n$ can be described by stereographic projection of a hyperplane intersection of the quadric $\mathcal{D}^{\sim}$ in $\mathbf{P}^3 = \mathbf{P}^{n+1}$ by formula (4.7). The above inversion $\alpha(\mathrm{T}, \mathrm{A}_1, \mathrm{x}_1, \mathrm{B}_1)$ of $\mathbf{P}^2 = \mathbf{P}^n$ can be described as a reflection of $\mathcal{D}^{\sim}$ in $\mathbf{P}^3 = \mathbf{P}^{n+1}$ as a cycle (hypersphere) preserving mapping, derived by stereographic projection from the above mapping $\mathbf{P}^2 = \mathbf{P}^n \to \mathcal{D}^{\sim}$ in $\mathbf{P}^3 = \mathbf{P}^{n+1}$ by formula (4.8). □

Acknowledgment. I thank my son, Zsolt Molnár for constructing the figures and my colleague, Dr. Jenő Szirmai for his help in preparing this manuscript!

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