FRÉGIER POINTS REVISITED*

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ABSTRAKT. Given a conic c and a point O on c, then the hypotenuses of right-angled triangles inscribed to c and having common vertex P intersect in one single point F, the Frégier point to O with respect to c. In the following this property will be referred to as "Theorem 1" and it is the starting point for further investigation. Varying O on c results in a point set $\{F\}$ of a curve f of 2nd order. In general, f is a conic concentric and similar to c. In some sense, Frégier's point of view is a variation of Thales' theorem, and it can therefore also be applied to the Theorem of the Angle at Circumference as well as to higher-dimensional analogues of the mentioned Frégier points and conics. Even so the topic roots in Projective Geometry, an elementary geometric treatment might also be of interest and suites also as Maths course material for high schools.

INTRODUCTION: THE FRÉGIER POINT THEOREM

About in 1815 M. Frégier published the in the following once more formulated Theorem 1, as mentioned in the abstract and visualized in Fig. 1. Wikipedia reveals nothing about him, references to Theorem 1 seem either to re-discover the result or are based on oral history. One can find theorem 1 e.g. in [4].

Theorem 1: All right-angled triangles inscribed to a conic s.t. their right-angle vertices coincide have hypotenuses passing through one point, the Frégier point.

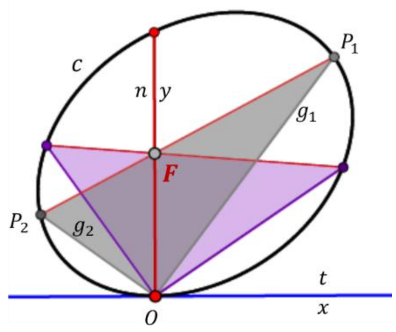

FIGURE 1: Hypotenuses of right-angled triangles inscribed to a conic in the Frégier point F.

There is an elegant geometric proof due to *H.-P. Schröcker* (JGG vol. 22/1) using a perspective collineation of the conic to a circle. There Frégier‘s theorem occurs as the classical Thales Theorem!

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The analytic proof is an easy exercise in Elementary Analytic Geometry:

$$c \ldots\ x^2 + a_1xy + a_2y^2 + a_3y = 0\,, \quad (a_3 \neq 0)$$
$$g_1 \ldots y = kx, \qquad g_2 \ldots\ y = -\frac{1}{k}x, \quad (k \in \mathbb{R})\,,$$
$$g_1 \cap c =: P_1 \quad \ldots \quad x_1 = -\frac{ka_3}{1+ka_1+k^2a_2}\,, \ \ y_1 = -\frac{k^2a_3}{1+ka_1+k^2a_2}$$
$$g_2 \cap c =: P_2 \quad \ldots \quad x_2 = +\frac{ka_3}{k^2-ka_1+a_2}\,, \ \ y_2 = -\frac{a_3}{k^2-ka_1+a_2}$$

Therewith follows for the slope of the hypotenuse

$$k_h := \frac{y_1-y_2}{x_1-x_2} = -\frac{k^2-ka_1-1}{k(1+a_2)} \quad ,$$

and finally, for the intersection of the hypotenuse line with the y-axis we get

$$y = k_h + d \ \ \Rightarrow \cdots\ d = -\frac{a_3}{1+a_2} \quad ,$$

what indeed is independent of the chosen slope *k*.

Remark: Theorem 1 is trivial from the projective geometric viewpoint: The pairs of legs of the right-angled triangles project the absolute involution at the line at infinity onto the „Steiner conic“ *c*. Therefore, the Frégier point *F* is nothing but the involution centre of the involutoric projectivity induced in *c*.

Variation of point *O* along the given conic *c* results in a set of Frégier points {*F*} fulfilling a conic, too, see Fig. 2 and the following calculation performed for the case of *c* being an ellipse:

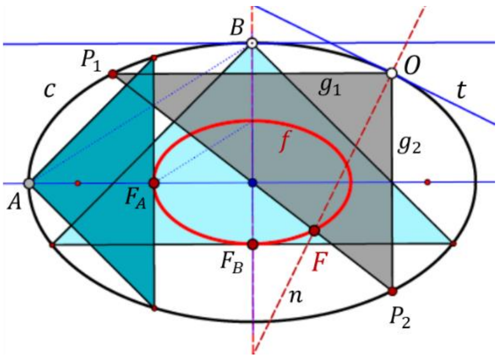

FIGURE. 2: Visualisation of a Frégier conic *f* to an ellipse *c*.

Choosing a point *O* arbitrarily on *c*, its Frégier point *F* is the intersection of two hypotenuses. The first one might be the one of the single degenerate right-angled triangle, the legs of which are tangent *t* and normal *n* of *c* at *O*, the latter also acting as hypotenuse. The second hypotenuse is a diameter of c, which connects the reflections P_1, P_2 of *O* at the axes of *c*. (For the vertices of *c* one must use another special second hypotenuse.) The following calculation is performed for an ellipse c, see Fig. 2, but for a parabola or a hyperbola c the calculation is similar.

$$c \dots O = \begin{pmatrix} a\cos\varphi \\ b\sin\varphi \end{pmatrix}, \ P_1 = \begin{pmatrix} -a\cos\varphi \\ b\sin\varphi \end{pmatrix}, \ P_2 = \begin{pmatrix} a\cos\varphi \\ -b\sin\varphi \end{pmatrix}.$$

$$P_1P_2 \dots y = -\frac{b}{a}\tan\varphi\, x, \qquad n \ \dots \ X = \begin{pmatrix} a\cos\varphi \\ b\sin\varphi \end{pmatrix} - t\begin{pmatrix} -b\cos\varphi \\ a\sin\varphi \end{pmatrix}.$$

$$F = n \cap P_1P_2 \ \Rightarrow \cdots \ t_F = \frac{2ab}{a^2+b^2}.$$

By replacing t by t_F one receives the parameter representation of an ellipse f concentric with c and with half axes

$$a_f = a\frac{a^2-b^2}{a^2+b^2}, \ b_f = b\frac{b^2-a^2}{a^2+b^2},$$

such that c and f are similar conics, but not centric similarly parameterised. Gathering these results we re-formulate the known

Theorem 2: The Frégier-points to points F of an ellipse c fulfil a concentric similar conic f, the “Frégier-conic” of c. The correspondence of the points O and F is not a centric similarity, as c and f are run through in opposite directions.

Theorem 2 holds also for parabolas and general hyperbolas c, see Fig. 3:

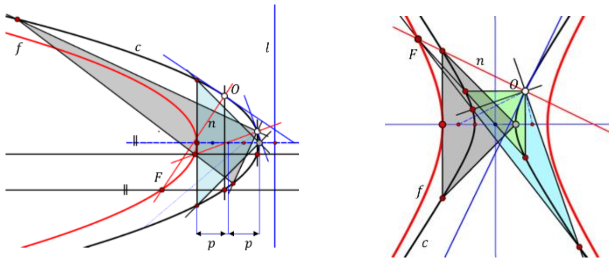

FIGURE. 3: The Frégier conic to a parabola c is an axial translate f and for a general hyperbola c it is a concentric and similar hyperbola f.

For a parabola c with parameter p the Frégier conic f is coaxial and congruent to c. Thereby the translation vector has length $2p$. For an equilateral hyperbola c the Frégier conic f degenerates to the line at infinity, and this property can be used to construct the normal n in an arbitrarily given point O of c, see Fig. 4.

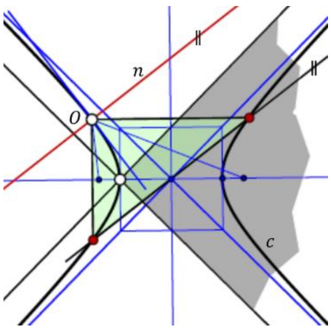

FIGURE. 4: For an equilateral hyperbola the Frégier conic is degenerated to the line at infinity.

1. THE FRÉGIER MAPPING, A QUADRATIC TRANSFORMATION

Place of action is the projective enclosed Euclidean plane $\mathbf{P}^2$. We declare the (planar) Frégier mapping γ: $\mathbf{P}^2 \to \mathbf{P}^2$ to fixed points O and F as follows:

Connect a given point P_1 with O, draw the line g_2 orthogonal to $OP_1 =: g_1$ through O and intersect g_2 with line $FP_1 =: f$. The intersection point $P_2 =: P_1^\gamma$ acts as Frégier image of P_1 see Fig. 5. Let P_1 run through a line l, then $f \to g_1$ is a perspectivity. As $g_1 \to g_2$ is an involutoric projectivity, the mappingt $f \to g_2$ is at least a projectivity. It is well-known that projective pencils $\{g_2\}$ and $\{f\}$ generate a conic l^γ. This in mind we can already recognise γ as a quadratic transformation. As P_2 maps to P_1 again, γ obviously is involutoric.

We base the analytic description of γ on a cartesian coordinate frame and use homogeneous coordinates:

$$O = (1,0,0)\mathbb{R},\ F = (1,0,1)\mathbb{R},\ \ P_1 = (1, x_1, y_1)\mathbb{R},\ P_2 = (1, x_2, y_2)\mathbb{R}\,.$$

The functions $x_2(x_1,y_1)$ and $y_2(x_1,y_1)$ follow from

$$f = P_1P_2 \ldots y = k_f x + 1,\ \ k_f = \frac{y_1 - 1}{x_1}, \qquad P_2 := f \cap g_2,$$

$$g_2 \ldots y = -\frac{x_1}{y_1}x, \quad \Rightarrow \cdots x_2 = \frac{-x_1y_1}{x_1^2+y_1^2-y_1},\ \ y_2 = \frac{x_1^2}{x_1^2+y_1^2-y_1},$$

and we can write the coordinate representation of γ as

$$\gamma: (1, x_1, y_2)\mathbb{R} \mapsto (x_1^2 + y_1^2 - y_1, -x_1y_1, x_1^2)\mathbb{R}\ .$$

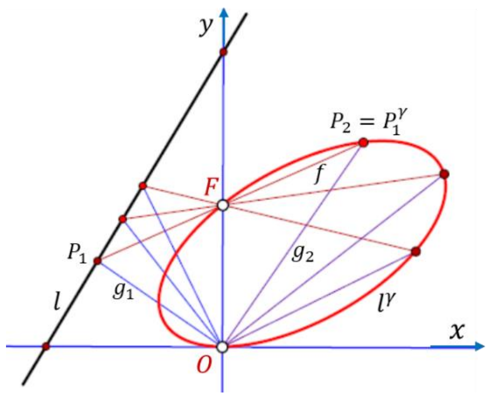

FIGURE. 5: The Frégier map of a general line l is a conic lγ .

The singularity figure of the involutoric quadratic mapping γ consists of the points

$$\left.\begin{array}{l} O = (1,0,0)\mathbb{R} \mapsto (0,0,0)\mathbb{R} \\ F = (1,0,1)\mathbb{R} \mapsto (0,0,0)\mathbb{R} \end{array}\right\}$$

and the coordinate lines only, as

$$\begin{array}{l} X = (1, x \neq 0, 0)\mathbb{R} \ \mapsto F = (1,0,1)\mathbb{R} \\ Y = (1, 0, y \neq 0, 1)\mathbb{R} \mapsto O = (1,0,0)\mathbb{R}\,. \end{array}$$

Therefore we can speak of a degenerate singularity triangle of the quadratic mapping γ, where two sides collapse to the line OF, and the third side is the line orthogonal to OF , (these lines are the y- and x-axis of the used frame). As γ is involutoric, we might speak of a "*Frégier inversion*".

Lines remaining fixed under γ belong to the pencils $O(...g_1...)$ and $F(...f...)$. The line u at infinity maps to the Thales circle u^γ over the segment $[O,F]$. There are exactly two fixed points of γ, namely the conjugate imaginary points $P_1 = P_2 = (1, \pm i, 1)\mathbb{R}$.

Collecting these results we state

Theorem 3: The Frégier-mapping γ is an involutoric quadratic mapping with degenerate singularity figure $\{O, F; y := OF,\ x\,(x \perp y,\ O \in x)\}$. The imaginary points $(1, \pm i, 1)\mathbb{R} \in l \perp OF,\ (F \in l)$ are the only fixed points of γ.

2. A "FRÉGIER ANGLE–AT–CIRCUMFERENCE THEOREM"

For a circle $\boldsymbol{c}$ yields: Inscribed triangles with same angle α at common vertex $O \in c$ have sides a (opposite to O) of equal length. These sides envelop a circle c_α concentric with c, see Fig. 6. This is a consequence of the classical angle-at-circumference theorem considering inscribed triangles with fixed base side.

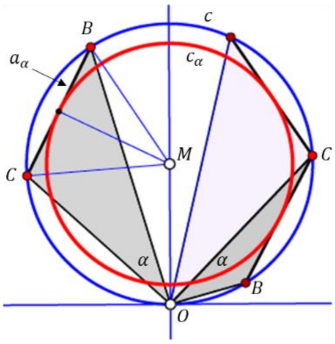

FIGURE. 6: Visualisation of a Frégier Angle at Circumference Theorem

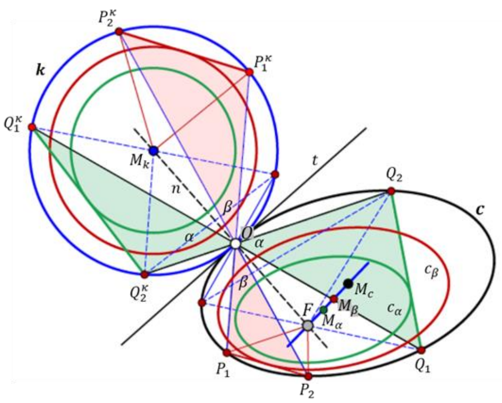

FIGURE. 7: Sketch of a proof for a "Frégier angle-at-circumference theorem".

For a conic c we proceed modifying an idea of H.P. Schröcker [2]: We consider sets of triangles with vertex O on c having equal angles α resp. β. A suitably chosen perspective collineation κ with centre O shall map c to a circle $k = c^\kappa$. The triangles are then mapped to ones inscribed to k, which have base sides a_α resp. a_β of equal lengths. These base sides envelop circles $c_\alpha{}^\kappa$ resp. $c_\beta{}^\kappa$, which must origin from conics c_α resp. c_β, see Fig. 7. As k, $c_\alpha{}^\kappa$ and. $c_\beta{}^\kappa$ belong to a (concentric) pencil of circles, also c, c_α and c_β belong to a pencil of conics. It turns out that c, c_α and c_β are homothetic conics and their centres are collinear with the Frégier point F to O with respect to c.

Theorem 4: Triangles inscribed to a conic c having equal angles α at common vertex $O \in c$ have sides a (opposite to O) enveloping a conic c_α. This conic is centric similar to c with respect to the Frégier point F to O.

Here again one might ask for the set of conics c_α,(O) for O varying on c and for triangles T with the same angle α at O, see Fig.8. We present the result without further arguments or proofs:

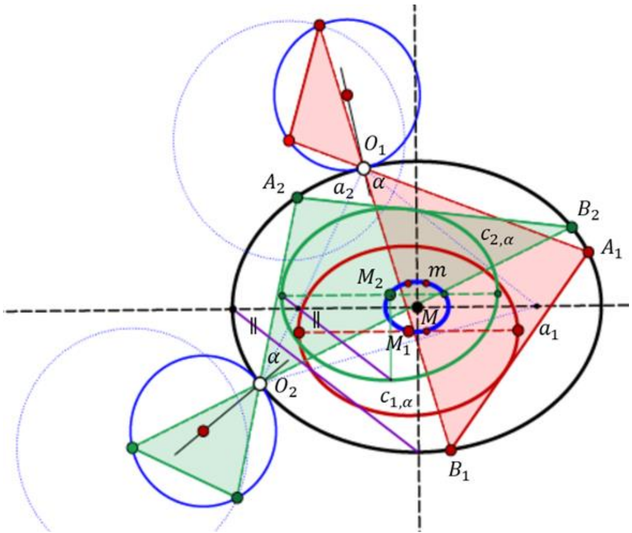

FIGURE. 8: The envelops $c_{i\alpha}$ of base lines a_i of triangles $T(\alpha, O_i)$, i = 1,2, are congruent conics with parallel axes.

Theorem 5: The envelops $c_{i\alpha}$ of base lines a_i of triangles $T(\alpha, O_i)$, i = 1,2,are congruent conics and homothetic with c. Their centres M_i belong to a "midpoint conic" m, which is co-axial with c and similar to c. All envelop conics to triangles $T(\alpha, O \in c)$ have a pair of envelop conics e, e' which again are co-axial with c and similar to c. For $\alpha \to \pi/2$ the conics $c_{i\alpha}$ degenerate to Frégier points, while m, e and e' collapse into the Frégier conic f of c.

3. 3D-VERSION OF THE FRÉGIER THEOREM 1

We generalise the right angle hook by three pairwise orthogonal lines a, b, c through the origin O of a cartesian 3-frame $\{O, x, y, z\}$ and intersect them with a quadric Φ touching the xy-plane at O, see Fig. 9.

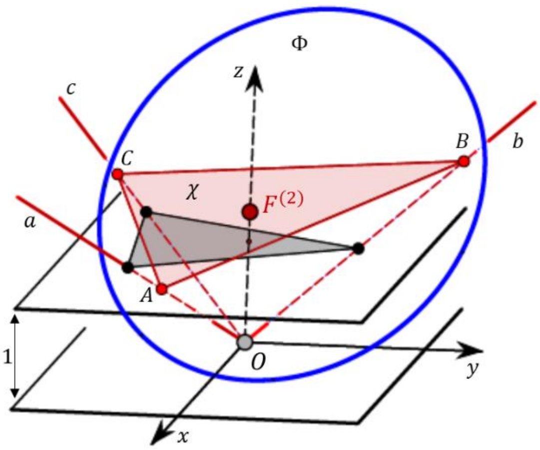

FIGURE 9: Sketch of a 3D-version of Frégier's Theorem 1.

Without loss of generalisation we can assume

$$\Phi \dots x^2 + a_1 y^2 + a_2 z^2 + a_3 xz + a_4 yz + a_5 z = 0,$$
$$a \dots \vec{a} = t_1 \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}, b \dots \vec{b} = t_2 \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}, c \dots \vec{c} = t_3 \begin{pmatrix} x_3 \\ y_3 \\ 1 \end{pmatrix}$$

By the orthogonality conditions we describe the coordinates of legs b and c as functions of x_1, y_1, x_2 and calculate the parameter values t_i of the intersection points A,B,C of a,b,c with Φ :

$$\vec{a}\vec{b} = \vec{b}\vec{c} = \vec{c}\vec{a} = 0. \implies$$

$$y_2 = y_2(x_1, y_1, x_2), y_3 = y_3(x_1, y_1, x_2), x_3 = x_3(x_1, y_1, x_2)$$

$$a \cap \Phi = A \implies t_1,\ b \cap \Phi = B \implies t_2, c \cap \Phi = C \implies t_3,$$

$$t_i = \frac{-a_5}{x_i^2 + a_1 y_i^2 + a_3 x_i + a_4 y_i + a_2}.$$

Finally we calculate the equation of the "hypotenuse plane" *ABC*

$$ABC = \chi \dots det \begin{vmatrix} 1 & x & y & z \\ 1 & t_1 x_1 & t_1 y_1 & t_1 \\ 1 & t_2 x_2 & t_2 y_2 & t_2 \\ 1 & t_3 x_3 & t_3 y_3 & t_3 \end{vmatrix} = 0\,,$$

(variable parameters are marked red; the other coordinates in this determinant are functions of these parameters). Using abbreviation symbols for the sub-determinants we can write the equation of χ as

$$\chi \dots .1. |1,2,3| - x. |0,2,3| + y. |0,1,3| - z. |0,1,2| = 0\ .$$

As we are interested in the intersection of χ with the z-axis of the frame, we need only the first and last sub-determinants to calculate the point F. After a lengthy calculation we receive

$$z_F := \frac{|1,2,3|}{|0,1,2|} = \ldots !!! \ldots \Longrightarrow \quad z_F = \frac{-a_5}{1+a_1+a_2}\ ,$$

which shows indeed the independence of the starting frame $\{O;a,b,c\}$.

Theorem 6: Orthoschemes $\{O; A, B, C\}$ inscribed into a quadric Φ having common vertex $O \in \Phi$ possess hypotenuse planes $\chi = ABC$, which intersect the normal n of Φ at O in the „Frégier point“ $F^{(2)}$ to O.

As in the 2D-case we look for the set of Frégier points $F^{(2)}$, if O moves on the quadric Φ. Now Schröcker's idea using a suitable perspective collineation κ transforming Φ to a sphere cannot be used any more: The (real or imaginary) generators of Φ at O would remain fixed and are, in general, not the isotropic generators of a sphere touching Φ at O.

Obviously $\{F^{(2)}\}$ has the same symmetries, as $\{F^{(2)}\}$. It turns out that $\{F^{(2)}\}$ is a quadric concentric with Φ, but not similar to Φ, see Fig. 10. We content ourselves with presenting only the result without proof as

Theorem 7: The set of Frégier-points $F^{(2)}$ to a quadric Φ is – in case of an ellipsoid and a (general) hyperboloid – a quadric concentric and coaxial with Φ, in case of a paraboloid Φ a coaxial paraboloid.

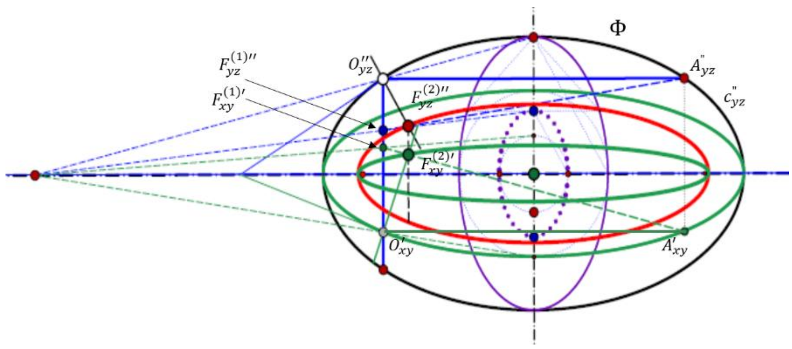

FIGURE 10: Sketch of the Frégier quadric **$\{F^{(2)}\}$** to an ellipsoid Φ (black/green/ purple ellipses…front/top/side view of Φ ; red/green/dashed purple …front/top/side view of **$\{F^{(2)}\}$).**

CONCLUSION

Omitting the discussion of special cases we showed some generalisations of Frégier's theorems 1 and 2. For (planar) non-Euclidean versions see [2]. We conjecture that the presented results also hold for higher dimensions. The calculation for dimension 3 is already very laborious. The special case of Φ being a sphere or an n-hypersphere is treated in [3], but there are still open questions concerning a 3D-version of an angle-at-circumference theorem. There is a special version of the angle-at-circumference theorem described in [1] as a generalisation of the classical version, which differs from the Frégier point of view, which we aimed at in this paper.

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