ANCIENT PROBLEMS VS. MODERN TECHNOLOGY

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Abstract. Geometric constructions using a ruler and a compass have been known for more than two thousand years. It has also been known for a long time that some problems cannot be solved using the ruler-and-compass method (squaring the circle, angle trisection); on the other hand, there are other problems that are yet to be solved.

Nowadays, the focus of researchers' interest is different: the search for new geometric constructions has shifted to the field of recreational mathematics.

In this article, we present the solutions of several construction problems which were discovered with the help of a computer.

The aim of this article is to point out that computer availability and performance have increased to such an extent that, today, anyone can solve problems that have remained unsolved for centuries.

1. Geometric constructions

Some problems, such as the search for a construction that would divide a given angle into three equal parts, the construction of a square having an area equal to the area of the given circle or doubling the cube, troubled mathematicians already hundreds and thousands of years ago. Today, we not only know that these problems have never been solved, but we are even able to prove that such constructions cannot exist at all [8], [10].

On the other hand, there is, for example, the problem of finding the center of a given circle with a compass alone. This is a problem that was admired by Napoleon Bonaparte [11] and one of the problems that we are able to solve today (Mascheroni found the answer long ago [9]).

As for other examples of the problems that we are now able to solve, there is the problem of finding the fraction of a given distance or constructing distances forming the ratio known as the Golden Ratio. The solutions to some of these tasks have been known for centuries, but new solutions and new constructions are still being discovered [6], [7].

How can constructions be discovered? And what do we actually mean by the word **construction**? Let us look at the following problem that is frequently solved by pupils during geometry lessons:

Given two points, find their midpoint using a ruler and a compass.

The possible solutions are numerous; in fact, their number is infinite. This is one of them:

(1) Draw a line connecting both points given.

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(2) Draw a circle with the center at the first point, passing through the second point.
(3) Draw a circle with the center at the second point, passing through the first point.
(4) Draw a line connecting the intersection points of both circles; the desired midpoint is its intersection with the first line.

Some kind of formal notation can be used to write down the sequence of steps of the constructions both briefly and precisely, for example (the construction above):

(1) $s_1 = AB$ (straight line s_1 passing through points A, B)
(2) $c_2 = A(B)$ (circle c_2 with the center A, passing through the point B)
(3) $c_3 = B(A)$, $\{C, D\} = c_2 \cap c_3$
(4) $s_4 = CD$, $\{P\} = s_1 \cap s_4$

So we have the **process** and its result – a **construction** as a set of geometrical objects (lines, circles, points).

2. How can we search for constructions? How many of them are there?

The elements of Euclidean geometric constructions are lines, circles and points. If some elements are given, there exists only a limited number of steps starting from these elements.

For example, given two points, A and B, we can construct a line passing through these points, or a circle with its center at one of the points and passing through the other point. Consequently, every construction based on two given points **A**, **B**, must start with the step **AB**, **A(B)** or **B(A)**.

Since none of these three steps have produced a new point, the next step must again be one of the same three steps. Consequently, after two steps of a construction starting from two given points, we have a line and one of the two circles or the two circles – see Fig. 1.

Any two lines, whether straight lines or circles, can intersect at a maximum of two points (they, of course, may not intersect at all) and any new point can be used for constructing a new straight line or a new circle.

We can continue this way and gradually discover all possible constructions starting from the given elements. However, the number of possible constructions is growing very quickly; as for constructions from two given points, it grows as follows:

$$3 - 3 - 19 - 359 - 22{,}192 - 8{,}708{,}346 - 20{,}115{,}793{,}428 - \dots$$

3. Old problems, new technology, results

Assuming that an ancient geometrician with a ruler and a compass would have needed 10 seconds to draw and examine one construction, the exploration of all constructions with seven lines would have taken him 201,157,934,280 seconds, which is more than 6,000 years!

Nevertheless, the amounts of constructions that are unmanageable for people can be managed by a computer and its programs. If using a program that can systematically search for all constructions, from the shortest one to longer ones and even longer ones, we can find out, for example,

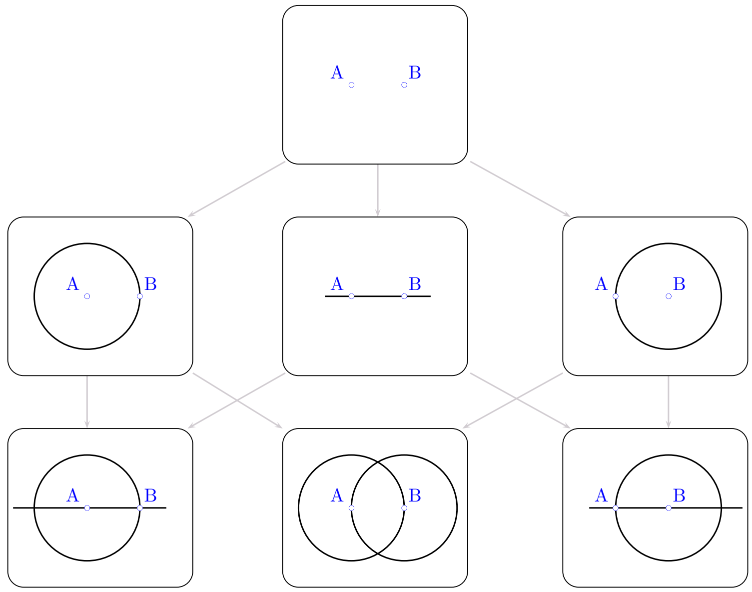

FIGURE 1. The first steps in constructions starting from two points

- that the golden ratio can be constructed with four lines using a ruler and a compass, and that there are only four different ways to do this;
- whereas, if we use a compass alone, we need five circles – see Fig. 2 – and there exist 368 different constructions [2];
- that six lines (straight lines or circles) are sufficient to find the k-th part of any distance for any k from five to twenty, except for one sixth...
- ...which can be constructed in five steps;
- whereas the construction of one 113th requires eight lines [1] – see Fig. 3;
- or that the construction of the right angle (i.e., a triple of points that form the vertex of the angle and the two points on the sides of the angle) using a compass alone requires drawing four circles – see Fig. 4.

This is how modern-day computers provide answers to questions that are hundreds and thousands of years old, such as in the case of the *four color theorem* [3].

4. How do we know all this?

It is easy to see all constructions starting from two given points which can be made in one step (by the term step, we understand one line: a straight line or a circle). The same holds for two-step constructions. For a larger number of steps, the situation is similar, but the number of constructions grows very quickly. We wrote a program (in the C# programming language) which is able to conduct a search for a larger and larger number of steps. The program is relatively simple, except for a few techniques preventing us from obtaining the same construction repeatedly in different ways. The source code is available from the authors on request.

In [2] we show four types of construction, using a ruler and a compass and using a compass without a ruler – in fact, there are two types of compasses, since the

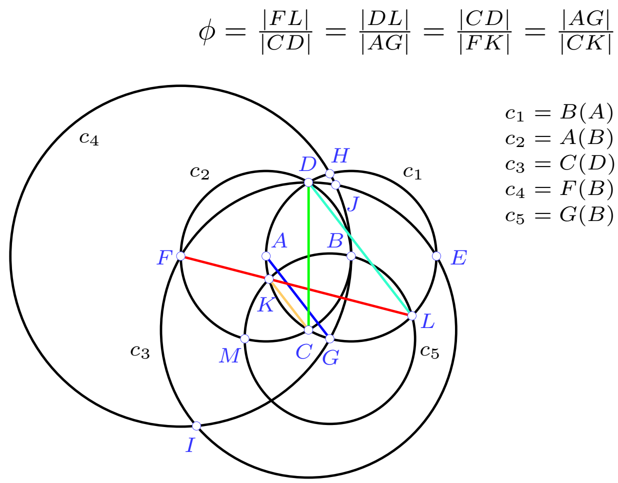

FIGURE 2. Construction No. CCO19 [2] of the golden ratio using five circles

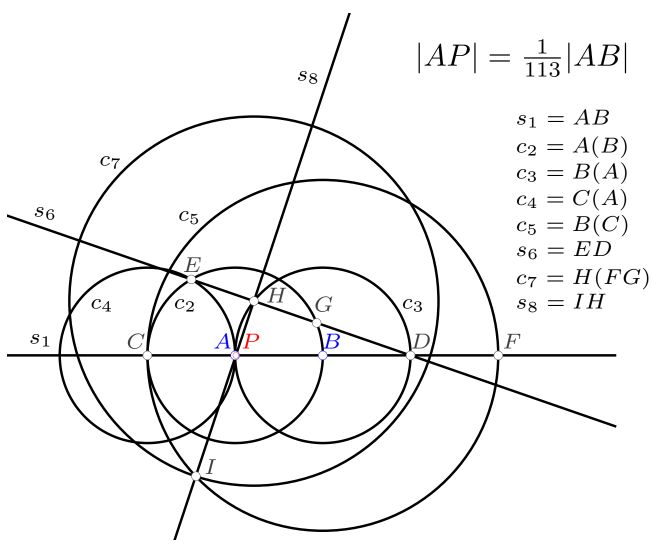

FIGURE 3. Construction of one 113th of a distance using eight lines

compass used by Euclid does not allow us to draw a circle with a given radius, only a circle passing through the given point.

Until now, we have searched for constructions of the golden ratio [2], of the k-th part of a distance [1], of the k-th part of the straight angle (both for integer k) and of some Mohr/Mascheroni constructions (without a ruler).

For example, Martin Gardner wrote in his Mathematical Games (cited from Mathematical Circus, [5]) that having only a compass, when starting from two given points on a diagonal, the missing two corners of a square can be constructed using

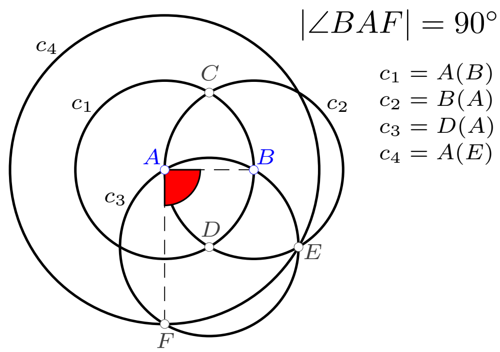

Figure 4. Construction of the right angle using four circles

nine circles. Later on, he presents a solution which is one step shorter, and, finally, a solution posted by his readers, which only requires six steps. The computer with its program allows us to say that this solution is the best one and that no shorter solution exists.

The computation of lines, circles and points was performed numerically – the program just eliminates the constructions that can not be correct. The next step is to prove that the specific construction is a solution of the given problem. That is the reason why parts of the above-mentioned book [2] or papers focus on proofs. Searching for all the seven-line constructions took 20 days using a standard PC; however, this task could be easily divided among multiple threads, cores and computers.

Conclusion

The point of the paper is that ordinary computers are now powerful enough to solve problems that have remained unsolved for centuries.

We have shown one kind of such problems in this paper and we would like to inspire readers to think about what other unresolved problems (not just geometric ones) might be cracked with the help of today's computers (such as, for example, scheduling, engineering network design and similar problems). Such a topic could be an interesting point of discussion between teachers and students during informatics seminars.

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