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(IN)EQUALITIES IN A TRIANGLE PROVED BY SYNTHETIC WAY

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ABSTRACT. There are many (in)equalities between elements of a triangle, whose justification is in the realm of high school mathematics. The most common procedure of their proofs is based on manipulation of trigonometric functions, like sum and difference formulas. In the article we show how some of the (in)equalities is possible to prove synthetically – by means of geometry and without trigonometric manipulation.

INTRODUCTION

There are many (in)equalities between elements of a triangle. Many of them is possible to justify in the realm of high school mathematics and some occur in Mathematical Olympiads. The most common way of their justifications is based on manipulation of trigonometric functions. We denote such approach as algebraic (manipulation with symbols according to given rules). This approach has a drawback: it requires experience in the treatment with often complex relationships and perfect knowledge of sum and difference formulas of trigonometric functions. It is known that many of the (in)equalities is possible to reduce to a concrete geometric property of a general triangle, which can be proved in a complete synthetic way. This approach of justification we denote as 'geometric'. Its advantage is elegance, straightforwardness, and gain of better understanding.

It is obvious that there is no clear cut between 'algebraic' and 'geometric' approach in solving (in)equalities. For example, the sum and difference formula may be proved in a synthetic way. Despite it, there is a difference if a student proves (in)equality of a triangle as a consequence of its geometrical property or by long manipulation of trigonometric formulas. We define 'algebraic' approach as a method requiring sum or difference formulas or various inequalities (such as A-G inequality, Cauchy inequality, or rearrangement of an expression to the sum of squares). If we solve an (in)equality in a synthetic way, based on relationship between elements of a triangle, we denote this approach as 'geometric'. Let's show an example, illuminating the difference between algebraic and geometric approach.

Prove that in an arbitrary triangle

$$\sin\alpha + \sin\beta > \sin\gamma.$$

Algebraic solution:

We will consequently rearrange the inequality

$$\sin\alpha + \sin\beta > \sin(180° - \alpha - \beta),$$
$$\sin\alpha + \sin\beta > \sin(\alpha + \beta),$$
$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} > 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2},$$
$$\cos\frac{\alpha-\beta}{2} > \cos\frac{\alpha+\beta}{2}.$$

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The last inequality is evident, since the cosine function is even and in interval (0,90°) decreasing. Because all steps of the rearrangement of the inequality are reversible, the proof of the inequality is done.

Geometric solution:

We multiply the supposed inequality by a positive number $2R$, which denotes the diameter of the circumscribed circle of a triangle with angles α, β, γ. We get

$$2R\sin\alpha + 2R\sin\beta > 2R\sin\gamma.$$

But it is easy to show that

$$2R\sin\alpha = a,\quad 2R\sin\beta = b,\quad 2R\sin\gamma = c,$$

so we arrived at the triangle inequality theorem

$$a + b > c.$$

Again, all steps are reversible and the inequality is proved.

In this article, eleven problems dealing with (in)equalities of a triangle are stated. The problems were chosen according to criterion whether the author was able to prove them by a geometric approach or not. As it was already stated, the aim of the article is to avoid a manipulation of trigonometric functions during a solution of a problem, like sum or difference formulas. There is a drawback of this approach as the solver must have more extensive knowledge enabling him to interpret an abstract (in)equality as a concrete property of a triangle.

We will start with some most common relationships between elements of a triangle with an outline of their proof. The problems are followed with their solutions. It is recommended to the reader to think about the problems before reading the solution.

All inequalities used in this article are well known. The main inspiration for writing the article was the book [1]. Although some problems in this article were adopted from that book, their solutions presented here are different.

The article is mainly addressed to teachers of mathematics and high school students interested in mathematics.

1. Some relations between elements of a triangle

Firstly, we summarize a typical notation concerning elements of a triangle ABC:

α, β, γ − angles at the vertices A, B, C

R, ρ − radius of circumscribed / inscribed circle of the tringle

O, I − centre of circumscribed / inscribed circle of the triangle

ρ_A, ρ_B, ρ_C − radii of circles escribed to the triangle opposite to vertices A, B, C

I_A, I_B, I_C − centres of circles escribed to the triangle opposite to vertices A, B, C

X, Y, Z – the points of touch of sides a, b, c of with inscribed circle

X_A, Y_A, Z_A − the points of touch of escribed circle opposite to vertex A with sides a, b, c (similarly for remaining two escribed circles)

A_1, B_1, C_1 − centres of sides BC, CA, AB of the triangle

$s = \frac{a+b+c}{2}$ − half of perimeter of the triangle

Part of the notation is depicted in Fig. 1.

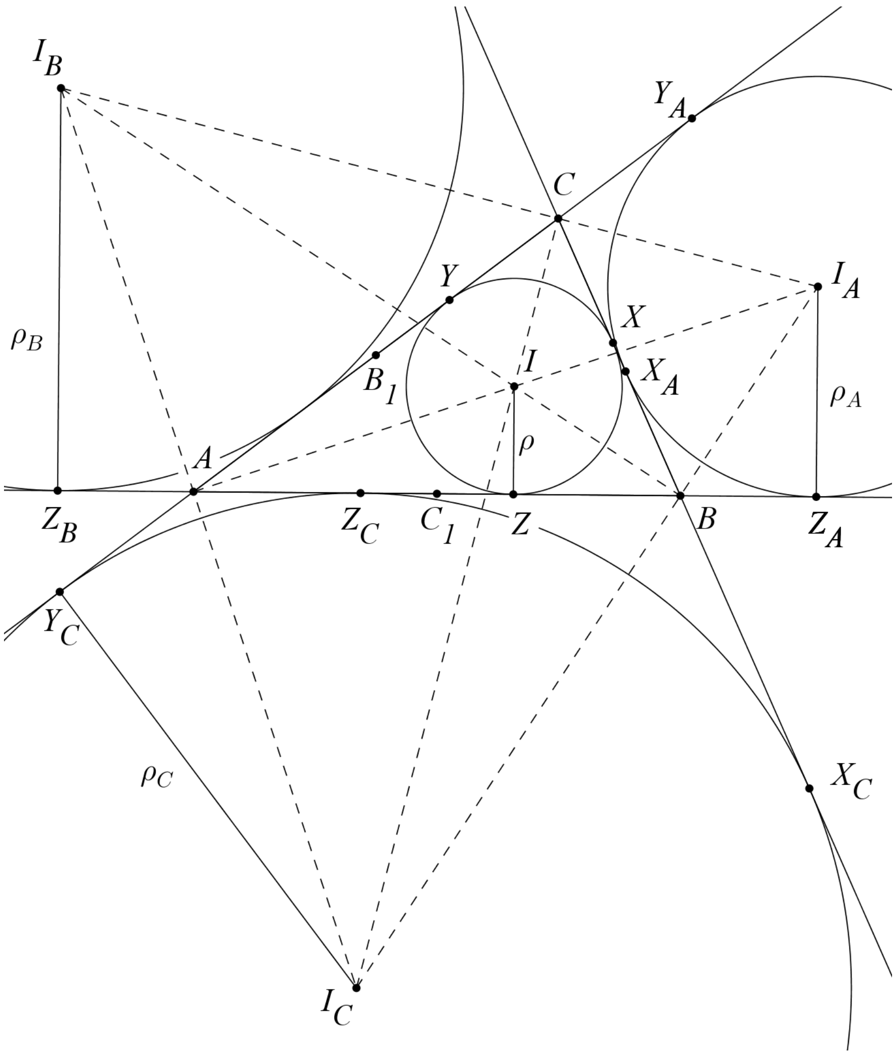


FIGURE 1. Illustration of the notation concerning elements of a triangle

We state here without proof some generally known facts:

$$|AZ_A| = |AY_A| = |BZ_B| = |BX_B| = |CY_C| = |CX_C| = \frac{a+b+c}{2} = s,$$
$$|AZ| = |AY| = s - a, \qquad |CX| = |CY| = s - c, \qquad |BZ| = |BX| = s - b.$$

Now we state two facts and four equations with proofs, which are critical for following considerations. All proofs are related to Fig. 2.

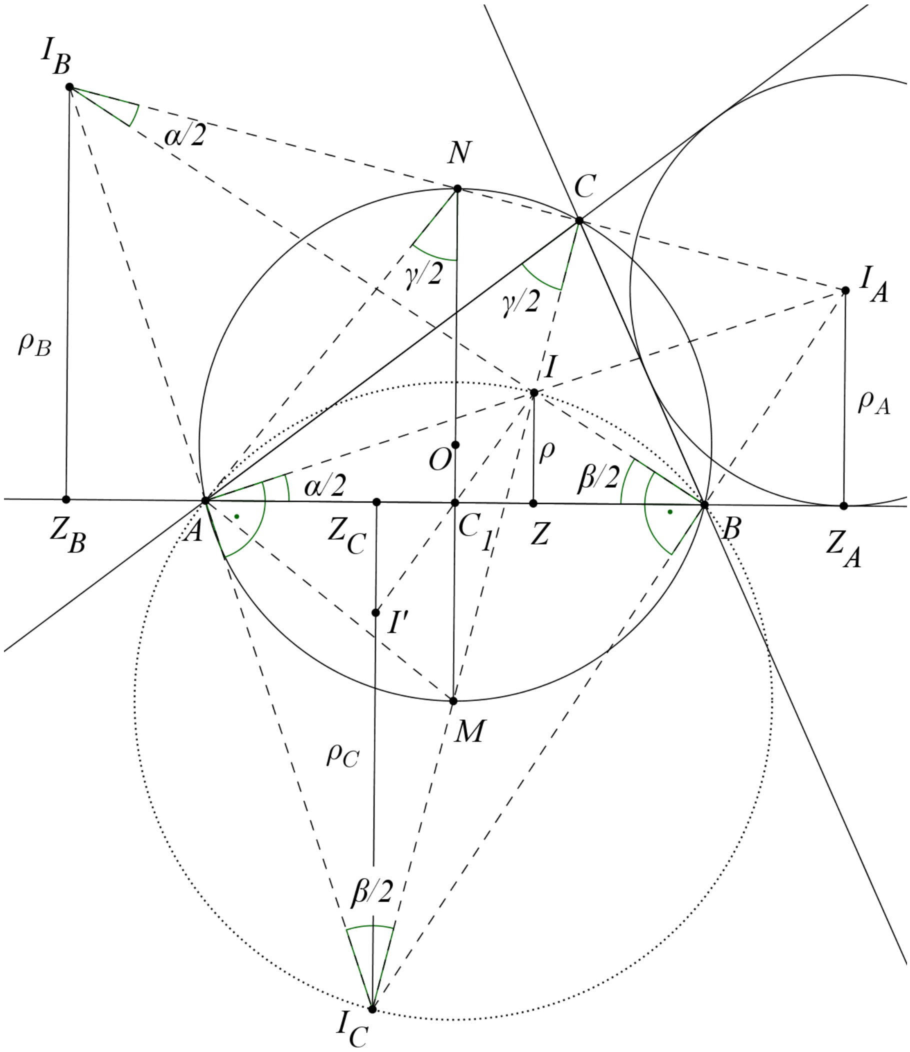


FIGURE 2. Proof of theorems 1.1 to 1.6.

Theorem 1.1.
The radius of the inscribed circle can be expressed as

$$\rho = 4R \cdot \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}. \tag{1}$$

Proof:
Let M be the middle point of the arc AB of the circle circumscribing the triangle ABC (Fig. 2). Then the following Lemma holds.

Lemma 1.2.
M is the centre of the circle circumscribed about quadrilateral $AIBI_C$.

Proof of Lemma 1.2. According to Thales' theorem, the centre of circumscribed circle about $AIBI_C$ lies on the segment II_C (the angles IAI_C and IBI_C are right, because they are made up of bisectors of adjacent angles). It must also lie on the perpendicular bisector of segment AB, i.e. on the line OM. As it is known, the lines intersect in the point M and the lemma is proved.

Proof of Theorem 1.1. We rearrange the right-hand side of the equation (1) in the following way:

$$4R \cdot \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2} = 2\left(2R \cdot \sin\frac{\gamma}{2}\right) \cdot \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} = 2|AM| \cdot \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2}$$
$$= |II_C| \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\alpha}{2} = |IA| \cdot \sin\frac{\alpha}{2} = \rho.$$

Theorem 1.3.
It is possible to express the half of perimeter of a triangle as
$$s = 4R \cdot \cos\tfrac{\alpha}{2} \cdot \cos\tfrac{\beta}{2} \cdot \cos\tfrac{\gamma}{2}. \qquad (2)$$

Proof:
Let N denotes the middle point of arc AB of circumscribed circle of the triangle containing the vertex C. Then the following Lemma holds.
Lemma 1.4.
The point N is the centre of the circle circumscribed about quadrilateral I_BABI_A.
Proof of Lemma 1.3. According to Thales' theorem, the centre of circumscribed circle about the quadrilateral I_BABI_A lies on the segment I_BI_A. It must also lie on the perpendicular bisector of segment AB, which is the line MN. It is known that the lines intersect in the point N. The Lemma is proved.
Proof of Theorem 1.3. The right-hand side of equation (2) can be expressed as
$$4R \cdot \cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2} = 2\left(2R \cdot \cos\frac{\gamma}{2}\right) \cdot \cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} = 2|AN| \cdot \cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2}$$
$$= |I_BI_A| \cdot \cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} = |I_BB| \cdot \cos\frac{\beta}{2} = |Z_BB| = s.$$

Theorem 1.5.
The following identity holds:
$$\tfrac{1}{2} \cdot (\rho_A + \rho_B + \rho_C - \rho) = 2R. \qquad (3)$$

Proof:
The quadrilateral $I_BZ_BZ_AI_A$ is trapezium and $|NC_1|$ is its midsegment (point N is the midpoint of segment I_BI_A and point C_1 is the midpoint of segment AB). Hence
$$|NC_1| = \frac{1}{2}(|I_BZ_B| + |I_AZ_A|) = \frac{1}{2}(\rho_A + \rho_B).$$
Moreover, since the point C_1 is the midpoint of segment Z_CZ (the proof is based on the fact that $|ZB| = s - b$ and $|AZ_C| = |AY_C| = |CY_C| - |CA| = s - b$), the symmetrical of the point I with respect to the centre C_1 is a point I' lying on the segment $|Z_CI_C|$. Let's consider a triangle $I'II_C$. It is evident, that
$$|I'I_C| = \rho_C - \rho$$
and
$$|MC_1| = \frac{1}{2}|I'I_C| = \frac{1}{2}(\rho_C - \rho).$$
Hence
$$2R = |MC_1| + |NC_1| = \frac{1}{2}(\rho_A + \rho_B) + \frac{1}{2}(\rho_C - \rho)$$
and the equation (3) is proved.

Theorem 1.6.
It holds
$$|OA_1| + |OB_1| + |OC_1| = R + \rho. \qquad (4)$$

Proof:
It is

$$|OC_1| = |OM| - |MC_1| = R - \frac{1}{2}(\rho_C - \rho).$$

Similarly, we express segments $|OA_1|$ and $|OB_1|$. Hence

$$|OA_1| + |OB_1| + |OC_1| = R + \rho + \left(2R - \frac{1}{2}\cdot(\rho_A + \rho_B + \rho_C - \rho)\right) = R + \rho.$$

Note: The relation

$$|OC_1| = R - \frac{1}{2}(\rho_C - \rho)$$

holds only in the case that the point C_1 lies between points O and M (in other words $|MC_1| \leq |MO|$). In the opposite case the distance $|OC_1|$ must be taken negative, because only under such convention the equation (4) holds in all cases. Since the relation

$$|OC_1| = R \cdot \cos\gamma$$

takes into account the sign (for $\gamma > 90°$ it gives $|OC_1|$ negative), we do not have to distinguish between acute and obtuse angled triangles.

Theorem 1.7.
Among all triangles inscribed in a given circle with radius R, the equilateral triangle has the greatest area and greatest perimeter. (The area is $S = R^2\frac{3\sqrt{3}}{4}$, the perimeter is $2s = 3\sqrt{3}R$.)

The proof is left to the reader.
(Hint: Use proof by contradiction. In the case of area, the task is trivial. In the case of perimeter, assume that at least two of the sides of the searched triangle are unequal and show that there is a triangle with greater perimeter. It is possible to solve both problems in a synthetic way.)

2. (IN)EQUALITIES IN A TRIANGLE PROVED BY 'GEOMETRIC APPROACH'

This section contains eleven problems on (in)equalities. Firstly, the problem is stated, then follows its solution. All solutions are in 'geometric fashion' (in the sense stated in introduction – the solution does not contain sum and difference formulas of trigonometric functions) and some of them rely on theorems formulated in the preceding section.
It is recommended to the reader to think about the problems (if she/he does not want to solve the problems) and only then to read the solution. The problems are arranged according to difficultness (which is a subjective criterion) and are partly interdependent.

(In)Equalities:

Problem 1.
Prove that

$$R \geq 2\rho.$$

(It is famous "Euler inequality")

Solution:
The simplest justification of this fact is following:

1. From all circles intersecting three sides of a triangle in (at least) one point, the circle inscribed in a triangle has the lowest radius.
2. The circle circumscribed about $A_1B_1C_1$ (circle passing through middle points of sides) has radius $R/2$.
3. Hence $\rho \leq \frac{R}{2}$, the equality occurring in the case of equilateral triangle

Problem 2.
Prove that

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha \cdot \sin\beta \cdot \sin\gamma.$$

Solution:
Let's multiply the equation by R^2:

$$R^2 \cdot \sin 2\alpha + R^2 \cdot \sin 2\beta + R^2 \cdot \sin 2\gamma = 4R^2 \cdot \sin\alpha \cdot \sin\beta \cdot \sin\gamma$$

Obviously $R^2 \cdot \sin 2\alpha = 2 \cdot S_{COB}$, where S_{COB} is area of triangle COB. Similarly, $R^2 \cdot \sin 2\beta = 2 \cdot S_{AOC}$ and $R^2 \cdot \sin 2\gamma = 2 \cdot S_{AOB}$, hence $R^2 \cdot \sin 2\alpha + R^2 \cdot \sin 2\beta + R^2 \cdot \sin 2\gamma = 2(S_{COB} + S_{AOC} + S_{AOB}) = 2 \cdot S_{ABC}$.
After rearranging the expression

$$4R^2 \cdot \sin\alpha \cdot \sin\beta \cdot \sin\gamma = (2R \cdot \sin\alpha)(2R \cdot \sin\beta) \cdot \sin\gamma = ab \cdot \sin\gamma = 2 \cdot S_{ABC}$$

the equality follows.

Problem 3.
Prove that

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin\alpha + \sin\beta + \sin\gamma.$$

Solution:
We will proceed as in the preceding case:

$$R^2 \cdot \sin 2\alpha + R^2 \cdot \sin 2\beta + R^2 \cdot \sin 2\gamma \leq R^2 \cdot \sin\alpha + R^2 \cdot \sin\beta + R^2 \cdot \sin\gamma$$

The left-hand side, as it is known, is equal $2 \cdot S_{ABC}$. Rearranging the right side, we get

$$R^2 \cdot \sin\alpha + R^2 \cdot \sin\beta + R^2 \cdot \sin\gamma = R(R\sin\alpha + R\sin\beta + R\sin\gamma) = R\left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right) = Rs,$$

where s is the half of perimeter.
On the basis of inequality $R \geq 2\rho$ (Problem 1) we obtain

$$R^2 \cdot \sin\alpha + R^2 \cdot \sin\beta + R^2 \cdot \sin\gamma = Rs \geq 2\rho s = 2 \cdot S_{ABC} =$$
$$= R^2 \cdot \sin 2\alpha + R^2 \cdot \sin 2\beta + R^2 \cdot \sin 2\gamma$$

Problem 4.
Prove that

$$\sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2} \leq \frac{1}{8}.$$

Solution:
Since $R \geq 2\rho$, we substitute from equation (1) to ρ and get

$$R \geq 8R \cdot \sin\alpha/2 \cdot \sin\beta/2 \cdot \sin\gamma/2$$

which is the proven inequality.

Problem 5.
Prove that

$$\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}.$$

Solution:
We have to prove

$$R\cos\alpha + R\cos\beta + R\cos\gamma \leq \frac{3}{2}R.$$

The left side of this inequality is

$$R\cos\alpha + R\cos\beta + R\cos\gamma = |OA_1| + |OB_1| + |OC_1|.$$

According to (4)

$$|OA_1| + |OB_1| + |OC_1| = R + \rho \leq R + \frac{R}{2} = \frac{3}{2}R.$$

Problem 6.

Prove that

$$\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$$

Solution:
We multiply the assumed inequality by R

$$R\sin\alpha + R\sin\beta + R\sin\gamma \leq R\frac{3\sqrt{3}}{2}$$

and interpret the left side

$$R\sin\alpha + R\sin\beta + R\sin\gamma = \frac{(a+b+c)}{2} = s.$$

According to theorem 1.7. the greatest perimeter among all triangles inscribed in a given circle has the equilateral triangle, hence $2s \leq 3\sqrt{3}R$ and the solution follows:

$$R\sin\alpha + R\sin\beta + R\sin\gamma = s \leq R\frac{3\sqrt{3}}{2}.$$

Problem 7.

Prove that

$$\sin\alpha \cdot \sin\beta \cdot \sin\gamma \leq \frac{3\sqrt{3}}{8}.$$

Solution:
We multiply the equation by $2R^2$

$$2R^2 \sin\alpha \cdot \sin\beta \cdot \sin\gamma \leq R^2\frac{3\sqrt{3}}{4}.$$

The left side is equal to S_{ABC}, as we derived in the problem 2,. Therefore, we have to prove

$$S_{ABC} \leq R^2\frac{3\sqrt{3}}{4}$$

which is (due to the Theorem 1.7) valid.

Problem 8.

Prove that

$$m_a + m_b + m_c \leq \frac{9}{2}R,$$

where m_a, m_b, m_c are medians of sides a, b, c of a triangle.

Solution:
From the triangle inequality theorem follows (Fig. 2)

$$m_c \leq |OC_1| + |OC| = |OC_1| + R$$

Similar inequalities are valid for other medians. Hence

$$m_a + m_b + m_c \leq 3R + |OA_1| + |OB_1| + |OC_1| = 4R + \rho \leq \frac{9}{2}R.$$

We used equation (4) and the conclusion of Problem 1.
Note: The reader should be aware of the fact, that the solution is valid only for right and acute-angled triangles. For obtuse-angled triangles, for example, with obtuse angle at the vertex C, the distance $|OC_1|$ would be negative (when using the equation (4)) and the proof collapses:

$$m_a + m_b + m_c \leq 3R + |OA_1| + |OB_1| + |OC_1| > 3R + |OA_1| + |OB_1| - |OC_1| = 4R + \rho.$$

It can be shown, that for obtuse-angled triangles a stronger inequality holds. However, the author was not able to prove it by purely 'geometric approach'. But it is possible to prove by synthetic way a certain "weaker theorem", from which the proof of the inequality follows: *To every obtuse-angled triangle, there exists another triangle, which has a greater sum of medians then the original one*. If there exists an obtuse-angled triangle, for which $m_a + m_b + m_c > \frac{9}{2}R$, then there must be an obtuse-angled triangle, for which the sum is maximal (among all triangles, since acute-angled triangles are eliminated). But this contradicts the "weaker theorem".
(Probably more elegant way how to eliminate obtuse-angled triangles exists. Alas, the author did not know it).

Problem 9.

Prove that

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\cdot\tan\frac{\alpha}{2} = 1.$$

Solution:
Dividing the supposed equality

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\cdot\tan\frac{\alpha}{2} = 1$$

by $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}$, we get an equivalent expression

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}.$$

Because (fig. 2)

$$\cot\frac{\alpha}{2} = \frac{|AZ|}{|IZ|} = \frac{s-a}{\rho}, \qquad \cot\frac{\beta}{2} = \frac{s-b}{\rho}, \qquad \cot\frac{\gamma}{2} = \frac{s-c}{\rho}$$

we can arrange the left side of the expression:

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \frac{s}{\rho}.$$

Finally, we replace s and ρ by formulas (1), (2):

$$\frac{s}{\rho} = \frac{4R\cdot\cos\alpha/2\cdot\cos\beta/2\cdot\cos\gamma/2}{4R\cdot\sin\alpha/2\cdot\sin\beta/2\cdot\sin\gamma/2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}.$$

Problem 10.

Prove that

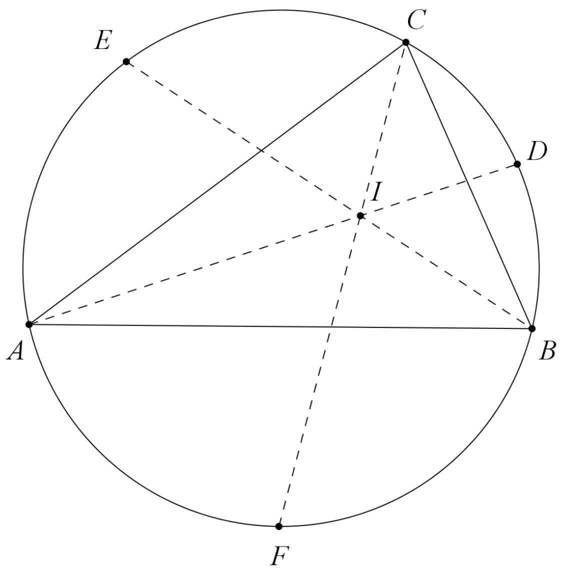
$$\frac{\rho_A}{h_a + 2\rho_A} + \frac{\rho_B}{h_b + 2\rho_B} + \frac{\rho_C}{h_c + 2\rho_C} = 1$$

where h_a, h_b and h_c are lengths of heights to the sides a, b, c.

Solution:
Because
$$S_{ABC} = S_{ACI_A} + S_{ABI_A} - S_{BCI_A} = \frac{1}{2}b\rho_A + \frac{1}{2}c\rho_A - \frac{1}{2}a\rho_A = \frac{b+c-a}{2}\rho_A = (s-a)\rho_A$$
(Fig. 2), it holds
$$\rho_A = \frac{S_{ABC}}{(s-a)}.$$
Further
$$h_a = \frac{2 \cdot S_{ABC}}{a}.$$
We rearrange the first term on the left side of the proven identity:
$$\frac{\rho_A}{h_a + 2\rho_A} = \frac{\frac{S_{ABC}}{(s-a)}}{\frac{2 \cdot S_{ABC}}{a} + \frac{2 \cdot S_{ABC}}{(s-a)}} = \frac{\frac{1}{(s-a)}}{\frac{2s}{a(s-a)}} = \frac{a}{2s}$$
Expressing similarly the remaining two terms, we get
$$\frac{a}{2s} + \frac{b}{2s} + \frac{c}{2s} = 1.$$

Problem 11.

Let I be the centre of the circle inscribed in a triangle ABC. Further notation is depicted on the following picture.



Prove that
$$|ID| \cdot |IE| \cdot |IF| \geq |IA| \cdot |IB| \cdot |IC|.$$

Solution:
We express the inequality in terms of radii R and ρ of circumscribed and inscribed circles.
Let's begin by the left-hand side. As follows from Lemma 1.2: $|IF| = |AF| = 2R\sin\frac{\gamma}{2}$.
Similarly, we express $|ID|$ and $|IE|$ and get

$$|ID|\cdot|IE|\cdot|IF| = 2R\sin\frac{\alpha}{2}\cdot 2R\sin\frac{\beta}{2}\cdot 2R\sin\frac{\gamma}{2} = 2R^2\left(4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right) = 2R^2\rho$$

(we used equation (1) in this expression).
For the right-hand side we have

$$|AI| = \frac{\rho}{\sin\frac{\alpha}{2}}.$$

Similarly, we express $|BI|$ and $|CI|$ and get

$$|IA|\cdot|IB|\cdot|IC| = \frac{\rho}{\sin\frac{\alpha}{2}}\cdot\frac{\rho}{\sin\frac{\beta}{2}}\cdot\frac{\rho}{\sin\frac{\gamma}{2}} = \frac{4R\cdot\rho^3}{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = 4R\cdot\rho^2.$$

The original inequality is equivalent to the inequality $2R^2\rho \geq 4R\cdot\rho^2$, or after simplification

$$R \geq 2\rho,$$

to ubiquitous Euler inequality.
The equality occurs only in the case of an equilateral triangle.

Conclusion

The purpose of the article is to show how some of (in)equalities in a triangle can be proven geometrically – by reducing given inequality to a property, which can be processed synthetically. Many publications are devoted to the topic of inequalities in geometry, e.g. the previously mentioned publication [1], which inspired the author to this article. It is needed to emphasize that not all inequalities can be solved purely geometrically (at least in a sense, that such approach is straightforward). As an illustration, here is an example with the solution demanding both geometrical and algebraic approach.

Example.
Prove that

$$(\rho_A+\rho_B)\cdot(\rho_B+\rho_C)\cdot(\rho_C+\rho_A) \leq 27R^3.$$

Solution:
According to A-G inequality:

$$(\rho_A+\rho_B)\cdot(\rho_B+\rho_C)\cdot(\rho_C+\rho_A) \leq \left(\frac{(\rho_A+\rho_B)+(\rho_B+\rho_C)+(\rho_C+\rho_A)}{3}\right)^3 =$$

$$= \left(\frac{2\cdot(\rho_A+\rho_B+\rho_C)}{3}\right)^3.$$

According to equation (3)

$$2\cdot(\rho_A+\rho_B+\rho_C) = 8R + 2\rho.$$

Finally, due to the conclusion of the Problem 1,

$$8R + 2\rho \leq 9R,$$

hence

$$\left(\frac{2\cdot(\rho_A+\rho_B+\rho_C)}{3}\right)^3 \leq \left(\frac{9R}{3}\right)^3 = 27R^3.$$

Publications usually do not distinguish between geometric and algebraic approaches. The purposeful task is to organize different approaches to geometric problems from pure synthetic to pure algebraic, or to solve identical problems in both ways. The article could serve as a first glance at this topic.

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