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EXISTENCE TO A BRINKMAN-LIKE MODEL FOR FLOWS THROUGH POROUS MEDIA

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Abstract. In this paper we study the existence of solutions to a certain model of flows through porous media. It is also shown that the model exhibits some regularity properties. We limit ourselves to the simplest case of periodic boundary conditions. This work has been directly motivated by the paper *K. R. Rajagopal*: On a hierarchy of approximate models for flows of incompressible fluids through porous solids. Math. Models Meth. Appl. Sci. **17** (2007), 215–252. The drag coefficient depends on pressure.

Introduction

We wish to investigate mathematical models describing flows through porous media in which the elevation of pressure has a substantial effect on the flow. We are interested in the well-posedness of such models that are listed and discussed from the viewpoint of the theory of mixtures in [7]. Here we treat the first non-elementary case where the interaction term between the fluid and the dispersed solid concerned has the form of the velocity difference multiplied with an interaction (friction) coefficient that depends on the local pressure. This is a natural non-linear generalization of the Brinkman model of porous media flows and is thermodynamically consistent (cf. [5]). We deal with a steady state system that can also be thought of as a system obtained after the time discretization by an implicit method suitable for a numerical solution. Similar problems were treated in [3], where the case of Darcy flows near an axisymmetric well with a pressure dependent drag coefficient is studied mostly numerically, [2], where the so called generalized Forchheimer law for slightly compressible fluids under various boundary conditions is investigated, and in [9], where the case of an instationary further generalized Forchheimer's law without pressure dependent permeability with a so-called pressure dependent distributed loss for slightly compressible flows is treated by the theory of maximal monotone operators.

Our paper is organized as follows. In Section 2, we set up notation and terminology. Further, we state the problem and introduce the notion of the weak solution. In Section 3, we proceed with the study of our problem

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and introduce two approximate systems whose solutions will converge to the solution of the original problem. In Section 4, we state the existence theorem. In Section 5, we obtain the apriori estimates and pass to the limit in the systems introduced in the third section and get a weak solution. In Section 6, we obtain some regularity of the weak solution which implies boundedness of pressure in the three-dimensional case. In the last Section, we comment on our results and set goals for future investigations.

1. Notation and Definitions

As we have already stated we deal with the case of periodic boundary conditions. Therefore we use the following notation:

- n — an integer greater of equal than 2, the space dimension (of the ambient space) $\mathbb{R}^n$
- i — an index running from 1 to n;
- L_i — a positive real number, the length of the edge of a $n-$dimensional parallelelpiped (Ω) along the i^{th} direction;
- Ω — a n–dimensional parallelepiped $\Omega = \prod_{i=1}^n (0, L_i)$ (a particular case of a bounded domain);
- $\mathbf{L}$ — the vector represented in the canonical Cartesian coordinates as $(L_1, \ldots, L_n)$;
- T — a positive real number, the length of a time interval $(0, T)$;
- Q_T — $Q_T = \Omega \times (0, T)$;
- ∇ — the gradient operator in Ω, $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$;
- div — the divergence operator in Ω, $\operatorname{div} \mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$, whenever this makes sense;
- Δ — Laplacian in Ω, $\Delta u = \operatorname{div} \nabla u$, whenever this makes sense.

We define standard differential operators of tensorial fields in the standard manner, i. e. componentwise using the definitions above. A tensor field S in called $\Omega - periodic$, iff $S(x_1 + L_1 k_1, x_2 + L_2 k_2, \ldots, x_n + L_n k_n) = S(x_1, x_2, \ldots, x_n)$ holds for all x and $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Our physical quantities we want to solve for are

- p — an $\Omega-$periodic scalar field $\Omega \to \mathbb{R}$, the pressure and
- $\mathbf{u}$ — an $\Omega-$periodic vector field $\Omega \to \mathbb{R}^n$, the velocity.

Here and subsequently we denote

- ν — a positive real number, the viscosity of the fluid;
- $\alpha(p)$ — a non-negative[1] real function, the "friction coefficient" between the fluid and solid phases;
- $\mathbf{f}$ — an $\Omega-$periodic vector field $\Omega \to \mathbb{R}^n$, the (external) body force;
- τ — a (small) positive real number, the timestep;
- ϵ, K — positive real numbers, parameters of approximations.

For simplicity of notation with the idea of investigating a weak solution we define the following function spaces

- $C^\infty_{per}(\Omega)$ — $C^\infty_{per}(\Omega) := \{p \mid p$ is $\Omega-$periodic, p has bounded derivatives of an arbitrary order$\}$;

[1] Usually non-decreasing.

- $L_0^q(\Omega)$ — $L_0^q(\Omega) := \{p \mid p \in L_{loc}^q(\Omega),\, p$ is $\Omega-$periodic, $\int_\Omega p(y)\,dy = 0$ $\}$;
- $\left[W_{per}^{1,2}(\Omega)\right]^n$ — $\left[W_{per}^{1,2}(\Omega)\right]^n := \overline{\{\mathbf{u} \mid \mathbf{u} \in \left[C_{per}^\infty(\Omega)\right]^n\}}$;
- $\left[W_{div,per}^{1,2}(\Omega)\right]^n$ — $\left[W_{div,per}^{1,2}(\Omega)\right]^n := \overline{\{\mathbf{u} \mid \mathbf{u} \in \left[C_{per}^\infty(\Omega)\right]^n, \operatorname{div}\mathbf{u} = 0\}}$
  where the closures are taken in the Sobolev $W^{1,2}(\Omega)-$norm.

Furthermore we denote by $(\cdot,\cdot)$ the scalar product in the space $L^2(\Omega)$. The same notation will be used for tensorial fields with components in $L^2(\Omega)$ without confusion. The duality between the space $\left[W_{per}^{1,2}(\Omega)\right]^n$ and its dual $\left(\left[W_{per}^{1,2}(\Omega)\right]^n\right)^*$ will be denoted by $\langle\cdot,\cdot\rangle$.

The standard truncation function is

$$T_K(t) = \begin{cases} t & |t| \le K \\ K & t > K \\ -K & t < -K \end{cases}$$

and it gives rise to a truncation operator for scalar fields in a natural way.

Let us denote by $||\mathbf{L}||^2$ the square of the diagonal in the body $\Omega : \sum_{i=1}^n L_i^2$.

For the right hand side of our system we will make use of the Helmholtz decomposition — any $\mathbf{f} \in L^p(\Omega)$ may be written as $\mathbf{f} = \mathbf{f}_1 + \nabla f_2$ where f_1 has zero distributional divergence and $f_2 \in L_{loc}^p(\Omega)$.

In the sequel we assume that the function α is Lipschitz continuous and that this growth condition holds

(A1) $\quad 0 \le \alpha(t) \le C(1+|t|^\gamma) \qquad C > 0\, \forall t \in \mathbb{R}$ and for a certain $\gamma < \infty$.

In the special bidimensional case we may relax the condition (A1) to
(A2)
$\alpha(t) \le C_3 + C_1 e^{C_2 t^\gamma} \qquad C_1, C_2, C_3 > 0\, \forall t \in \mathbb{R}$ and for a certain $\gamma < 2$.

For a further regularity of a weak solution we need also that $\alpha \in C^1(\mathbb{R})$ and

(A3) $\quad \left|\alpha'(t)\right| \le C(1+|t|^\gamma) \qquad C > 0\, \forall t \in \mathbb{R}$ and for a certain $\gamma < \infty$.

Remark 1.1. We might have assumed

$$\alpha(t) \ge C_\alpha > 0 \qquad \forall t \in \mathbb{R}$$

as well, but this condition is not really needed here.

1.1. **Statement of the problem.** The seminal work of Rajagopal [7] has introduced a whole hierarchy of models describing porous media flows in the context of mixture theory. They are relevant for situations like sand dispersed in oil. On the easiest level the solid is not modeled at all, because it is assumed that it is rigid (and it has a low volume fraction). In this case, the Galilean invariance principle is applied in order to consider the coordinate system connected with the solid which therefore remains at rest. This is naturally an approximation leading to a system of equations for the fluid only. The presence of the solid may be seen in the interaction term only. Here we assume a simple form of this term, namely that it depends on the velocity $\mathbf{u}$ in the linear way and on the pressure in a generally non-linear way. This is a generalization of the Brinkman model which assumes that the interaction does not depend on the pressure p. However, in some experiments pertinent to engineering practice such an assumption is not very realistic when pressure gradients are large enough (cf. reference in [7]).

In this paper, we neglect inertial terms and assume that we have discretized our system in time via an implicit method. We get

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p + \alpha(p)\mathbf{u} + \frac{1}{\tau}\mathbf{u} = \mathbf{f} + \frac{1}{\tau}\mathbf{w} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \\ \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega \\ \mathbf{u}, \mathbf{w}, p \quad \Omega - \text{periodic} \\ \int_\Omega p(x)\,dx = 0, \end{aligned} \tag{1.1}$$

where $\mathbf{w}$ denotes the velocity "at the previous time level" and is considered to be known.

For the existence of a solution to the system (1.1), we define the notion of the weak solution and in Section 3, we introduce two approximations to the system.

1.2. **Definition of the weak solution.** Without loss of generality we may redefine the functions $\alpha(t) := \alpha(t) + \frac{1}{\tau}$ and $\mathbf{f} := \mathbf{f} + \frac{1}{\tau}\mathbf{w}$ to get from (1.1) the equation

$$-\nu\Delta\mathbf{u} + \nabla p + \alpha(p)\mathbf{u} = \mathbf{f}. \tag{1.2}$$

Definition 1.2 (Definition of the weak solution to the system (1.1))**.** We call the pair $(\mathbf{u}, p) \in \left[W^{1,2}_{div,per}(\Omega)\right]^n \times L^q_0(\Omega)$ for all $q < \infty$ a weak solution to the system (1.1) iff

$$\nu\,(\nabla\mathbf{u}, \nabla\boldsymbol{\phi}) - (p, \operatorname{div}\boldsymbol{\phi}) + (\alpha(p)\mathbf{u}, \boldsymbol{\phi}) = \langle\mathbf{f}, \boldsymbol{\phi}\rangle \tag{1.3}$$

is satisfied for all vector fields $\boldsymbol{\phi}$ in $\left[W^{1,2}_{div,per}(\Omega)\right]^n$.

2. Approximations

In this section we introduce two approximations to the problem (1.1). The first one is defined with the aim of getting some approximative solutions converging to the weak solutions in the sense of the definition in Section 2. The second one helps us to pass to the limit in the non-linear term by a L^p-theory for pressure. Without it we would just have a similar L^2-theory which could have been used for convergence in the non-linear term for a rather limited range of the exponent γ from the assumption (A1).

The $\epsilon-$approximative system reads

$$\begin{aligned} -\nu\Delta\mathbf{u}^{K,\epsilon} + \nabla p^{K,\epsilon} + \alpha(T_K(p^{K,\epsilon}))\mathbf{u}^{K,\epsilon} = \mathbf{f} \quad \text{in } \Omega \\ -\epsilon\Delta p^{K,\epsilon} + \operatorname{div}\mathbf{u}^{K,\epsilon} = 0 \quad \text{in } \Omega \\ \mathbf{u}^{K,\epsilon}, p^{K,\epsilon} \quad \Omega - \text{periodic} \\ \int_\Omega p^{K,\epsilon}(x)\,dx = 0. \end{aligned} \tag{2.1}$$

This does not correspond to a usual slightly compressible fluid with a equation of state relating the pressure and the density. Instead, $((2.1)_2)$ is essentially a parabolic regularization of the continuity equation for isothermal, almost incompressible fluids. Incompressible fluids are described by the condition $(1.1)_2$ and the former equation has an advantage of resulting in a uniformly elliptic problem.

The truncated system is

$$\begin{aligned} -\nu\Delta\mathbf{u}^K + \nabla p^K + \alpha(T_K(p^K))\mathbf{u}^K &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u}^K &= 0 \quad \text{in } \Omega \\ \mathbf{u}^K, p^K &\quad \Omega-\text{periodic} \\ \int_\Omega p^K(x)\,dx &= 0. \end{aligned} \tag{2.2}$$

Definition 2.1 (Weak formulation of the approximative system (2.1))**.** We call the pair $(\mathbf{u}^{K,\epsilon}, p^{K,\epsilon}) \in \left[W^{1,2}_{per}(\Omega)\right]^n \times W^{1,2}_{per}(\Omega)$ a weak solution of the system (2.1), iff

$$\begin{aligned} \nu\left(\nabla\mathbf{u}^{K,\epsilon}, \nabla\boldsymbol{\phi}\right) - (p^{K,\epsilon}, \operatorname{div}\boldsymbol{\phi}) + (\alpha((T_K(p^{K,\epsilon})))\mathbf{u}^{K,\epsilon}, \boldsymbol{\phi}) &= \langle\mathbf{f}, \boldsymbol{\phi}\rangle \\ \epsilon(\nabla p^{K,\epsilon}, \nabla\psi) &= (\mathbf{u}^{K,\epsilon}, \nabla\psi) \end{aligned} \tag{2.3}$$

are satisfied for all vector fields $\boldsymbol{\phi}$ in $\left[W^{1,2}_{per}(\Omega)\right]^n$ and ψ in $W^{1,2}_{per}(\Omega)$.

Definition 2.2 (Weak formulation of the truncated system (2.2))**.** We call the pair $(\mathbf{u}^{K,\epsilon}, p^{K,\epsilon}) \in \left[W^{1,2}_{div,per}(\Omega)\right]^n \times L^2_{per}(\Omega)$ a weak solution of the system (2.2), iff

$$\nu\left(\nabla\mathbf{u}^K, \nabla\boldsymbol{\phi}\right) - (p^K, \operatorname{div}\boldsymbol{\phi}) + (\alpha((T_K(p^K)))\mathbf{u}^K, \boldsymbol{\phi}) = \langle\mathbf{f}, \boldsymbol{\phi}\rangle \tag{2.4}$$

are satisfied for all vector fields $\boldsymbol{\phi}$ in $\left[W^{1,2}_{per}(\Omega)\right]^n$.

3. Existence theorem

Theorem 3.1 (existence of a weak solution)**.** *For every non-negative* $\alpha \in C(\mathbb{R})$ *satisfying the growth conditions* (A1) *for* $n \geq 3$ *or* (A2) *for* $n = 2$ *and* ∇f_2 *in* $\left[L^n_{per}(\Omega)\right]^n$ *there exists a weak solution to the system* (1.1) $(\mathbf{u}, p) \in \left[W^{1,2}_{div,per}(\Omega)\right]^n \times L^q_0(\Omega)$ $\forall q < \infty$*. Moreover*

$$\frac{\nu}{2}||\nabla\mathbf{u}||_2^2 + \int_\Omega \alpha(p)|\mathbf{u}|^2\,dx \leq C \tag{EI}$$

holds.

Proof. The strategy of the proof consists in obtaining apriori estimates for the approximations, asserting existence of solutions to the $\epsilon-$approximation and passing to the limit with $\epsilon \to 0+$ and $K \to \infty$ by the Lebesgue dominated convergence theorem, resp. the Vitali convergence theorem. This is the content of the next section. The Vitali convergence theorem is essential in the proof and therefore is stated and discussed in some details. We omit the first step of the proof: to show that for a fixed $\epsilon > 0$ and $K > 0$

there exists a weak solution to the (2.1) in the sense of (2.3); this is very standard. □

4. Apriori estimates and passage to the limits

4.1. **Apriori estimates.** In this subsection, we want to derive mainly uniform bounds of the $\mathbf{u}^{K,\epsilon}$ and $p^{K,\epsilon}$ with respect to the parameters ϵ and K which will be used in the next subsection for the limit passage.

Let us start with "$\epsilon-$estimates". Note first that taking $\boldsymbol{\phi} = \mathbf{u}^{K,\epsilon}$ in $(2.3)_1$ and $\psi = p^{K,\epsilon}$ in $(2.3)_2$, using integration by parts $\left(p^{K,\epsilon}, \operatorname{div} \mathbf{u}^{K,\epsilon}\right) = -\left(\mathbf{u}^{K,\epsilon}, \nabla p^{K,\epsilon}\right)$, the Poincar inequality in the periodic setting, the Hölder and Young inequality we conclude

$$
(\mathrm{EI}_{K,\epsilon}) \quad \frac{\nu}{2}||\nabla \mathbf{u}^{K,\epsilon}||_2^2 + \int_\Omega \alpha((T_K(p^{K,\epsilon})))|\mathbf{u}^{K,\epsilon}|^2\,dx + \epsilon||\nabla p^{K,\epsilon}||_2^2 \leq \frac{2||\mathbf{L}||^2+\pi^2}{2\nu\pi^2}||\mathbf{f}||_{-1,2}^2 + \frac{\nu\pi^2}{2\left(2||\mathbf{L}||^2+\pi^2\right)}\left(\prod_{i=1}^n L_i\right)^2,
$$

being an analogon of (EI). We observe that $(\mathrm{EI}_{K,\epsilon})$ is a uniform bound for velocities $\mathbf{u}^{K,\epsilon}$, but a non-uniform one for pressures $p^{K,\epsilon}$. Since our system (2.3) is non-linear in the pressure, we generally need a uniform bound. To obtain it we take $\boldsymbol{\phi} = \nabla p^{K,\epsilon}$ in $(2.3)_1$ and treat it similarly as later in (4.4)–(4.7)

$$
\begin{aligned}
\nu\left(\nabla\mathbf{u}^{K,\epsilon}, \nabla\nabla p^{K,\epsilon}\right) - (p^{K,\epsilon}, \Delta p^{K,\epsilon}) + (\alpha((T_K(p^{K,\epsilon})))\mathbf{u}^{K,\epsilon}, \nabla p^{K,\epsilon}) &= \\
\langle \mathbf{f}, \nabla p^{K,\epsilon}\rangle& \\
-\nu\left(\nabla \mathrm{div}\ \mathbf{u}^{K,\epsilon}, \nabla p^{K,\epsilon}\right) + (\nabla p^{K,\epsilon}, \nabla p^{K,\epsilon}) - \left(\mathrm{div}\ \left(\alpha((T_K(p^{K,\epsilon})))\mathbf{u}^{K,\epsilon}\right), p^{K,\epsilon}\right) &= \\
\langle \mathbf{f}_1 + \nabla f_2, \nabla p^{K,\epsilon}\rangle& \\
-\nu\epsilon\left(\nabla\Delta p^{K,\epsilon}, \nabla p^{K,\epsilon}\right) + (\nabla p^{K,\epsilon}, \nabla p^{K,\epsilon}) - \left(\mathrm{div}\ \left(\alpha((T_K(p^{K,\epsilon})))\mathbf{u}^{K,\epsilon}\right), p^{K,\epsilon}\right) &= \\
\langle \mathrm{div}\ \mathbf{f}_1, \nabla p^{K,\epsilon}\rangle + \left(\nabla f_2, \nabla p^{K,\epsilon}\right)& \\
\nu\epsilon\left(\Delta p^{K,\epsilon}, \Delta p^{K,\epsilon}\right) + (\nabla p^{K,\epsilon}, \nabla p^{K,\epsilon}) &= \\
\left(\nabla f_2, \nabla p^{K,\epsilon}\right) + \left(\alpha((T_K(p^{K,\epsilon})))\mathrm{div}\ \mathbf{u}^{K,\epsilon} + \nabla\alpha((T_K(p^{K,\epsilon})))\cdot\mathbf{u}^{K,\epsilon}, p^{K,\epsilon}\right)& \\
\nu\epsilon\left(\Delta p^{K,\epsilon}, \Delta p^{K,\epsilon}\right) + (\nabla p^{K,\epsilon}, \nabla p^{K,\epsilon}) - \left(\nabla f_2, \nabla p^{K,\epsilon}\right) &= \\
\left(\epsilon\alpha((T_K(p^{K,\epsilon})))\Delta p^{K,\epsilon} + \alpha'((T_K(p^{K,\epsilon})))\xi_{[-K,K]}(p^{K,\epsilon})\cdot\mathbf{u}^{K,\epsilon}\nabla p^{K,\epsilon}, p^{K,\epsilon}\right)&
\end{aligned}
$$

$$\nu\epsilon\int_\Omega |\Delta p^{K,\epsilon}|^2\,dx + \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx - \left(\nabla f_2, \nabla p^{K,\epsilon}\right) =$$

$$+\int_\Omega \nabla H(p^{K,\epsilon})\cdot \mathbf{u}^{K,\epsilon}\,\mathrm{d}x + \epsilon\int_\Omega \alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\,dx$$

$$\nu\epsilon\int_\Omega |\Delta p^{K,\epsilon}|^2\,dx + \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx - \left(\nabla f_2, \nabla p^{K,\epsilon}\right) =$$

$$-\int_\Omega H(p^{K,\epsilon})\mathrm{div}\ \mathbf{u}^{K,\epsilon}\,\mathrm{d}x + \epsilon\int_\Omega \alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\,dx$$

$$\nu\epsilon\int_\Omega |\Delta p^{K,\epsilon}|^2\,dx + \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx - \left(\nabla f_2, \nabla p^{K,\epsilon}\right) =$$

$$+\epsilon\int_\Omega H(p^{K,\epsilon})\Delta p^{K,\epsilon}\,\mathrm{d}x + \epsilon\int_\Omega \alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\,dx$$

$$\nu\epsilon\int_\Omega |\Delta p^{K,\epsilon}|^2\,dx + \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx - \left(\nabla f_2, \nabla p^{K,\epsilon}\right) =$$

$$-\epsilon\int_\Omega H'(p^{K,\epsilon})\nabla p^{K,\epsilon}\cdot\nabla p^{K,\epsilon}\,\mathrm{d}x + \epsilon\int_\Omega \alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\,dx$$

$$\begin{aligned}\nu\epsilon\int_\Omega |\Delta p^{K,\epsilon}|^2\,dx + \epsilon\int_\Omega H'(p^{K,\epsilon})|\nabla p^{K,\epsilon}|^2\,dx + \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx =\\ \int_\Omega \nabla p^{K,\epsilon}\cdot\nabla f_2\,\mathrm{d}x + \epsilon\int_\Omega \alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\,\mathrm{d}x\end{aligned} \tag{4.1}$$

$$\begin{aligned}C(n,\mathbf{L})||p^{K,\epsilon}||^2_{\frac{2n}{n-2}} \le \int_\Omega |\nabla p^{K,\epsilon}|^2\,dx \le \int_\Omega |\nabla p^{K,\epsilon}\cdot\nabla f_2|\,\mathrm{d}x +\\ \epsilon\int_\Omega \left|\alpha((T_K(p^{K,\epsilon})))p^{K,\epsilon}\Delta p^{K,\epsilon}\right|\,\mathrm{d}x + \epsilon\int_\Omega \left|H'(p^{K,\epsilon})\right||\nabla p^{K,\epsilon}|^2\,\mathrm{d}x \le\\ \left(\epsilon K||\alpha'||_{L^\infty([-K,K])} + \epsilon||\alpha||_{L^\infty([-K,K])}\right)||\nabla p^{K,\epsilon}||^2_L(\Omega)^2 +\\ \frac{1}{2}||\nabla p^{K,\epsilon}||^2_{L^2(\Omega)} + \frac{1}{2}||\nabla f_2||^2_{L^2(\Omega)}\end{aligned}$$

Thus,

$$\begin{aligned}&C'(n,\mathbf{L})||p^{K,\epsilon}||^2_{\frac{2n}{n-2}} \le\\ &\int_\Omega |\nabla p^{K,\epsilon}|^2\,\mathrm{d}x \le \frac{||\nabla f_2||^2_{L^2(\Omega)}}{1-2\epsilon\left(K||\alpha'||_{L^\infty([-K,K])} + ||\alpha||_{L^\infty([-K,K])}\right)}.\end{aligned} \tag{4.2}$$

with $n > 2$, $C(n,\mathbf{L})$ a positive constant obtained from the Gagliardo-Nirenberg-Sobolev and Poincaré's inequalities, $H(t) :=$ $\int_0^t \alpha'(T_K(s))\xi_{[-K,K]}(s)s\,ds$ with ξ_B being the characteristic function of the set B by the Hölder and Cauchy-Schwarz inequalities.

Remark 4.1. The problem with non-differentiability in (4.1) does not pose any threat since we have got differentiability almost everywhere in Ω as the function $\alpha(t)$ is differentiable for almost all real t (if supposed to be non-decreasing which may be a suitable physical assumption or Lipschitz continuous), the truncation function $T_K(t)$ is differentiable for all real t except for two, and we dispose of a non-uniform estimate ($\mathrm{EI}_{K,\epsilon}$).

Next we proceed with "K-estimates". In the same manner as before we take $\boldsymbol{\phi} = \mathbf{u}^K$ in (2.4) using Poincaré and Young's inequalities getting

$$\nu\left(\nabla \mathbf{u}^K, \nabla \mathbf{u}^K\right) - (p^K, \operatorname{div} \mathbf{u}^K) + \int_\Omega \alpha((T_K(p^K)))|\mathbf{u}^K|^2 \,\mathrm{d}x = \langle \mathbf{f}, \mathbf{u}^K \rangle$$
$$\nu||\nabla \mathbf{u}^K||_2^2 + \int_\Omega \alpha((T_K(p^K)))|\mathbf{u}^K|^2 \,\mathrm{d}x \le ||\mathbf{f}||_{-1,2}||\mathbf{u}^K||_{1,2}$$
$$\nu||\nabla \mathbf{u}^K||_2^2 + \int_\Omega \alpha((T_K(p^K)))|\mathbf{u}^K|^2 \,\mathrm{d}x \le \frac{||\mathbf{L}||^2 + \pi^2}{2\nu\pi^2}||\mathbf{f}||_{-1,2}^2 + \frac{\nu}{2}||\nabla \mathbf{u}^K||_2^2$$

$$(\mathrm{EI}_K) \qquad \frac{\nu}{2}||\nabla \mathbf{u}^K||_2^2 + \int_\Omega \alpha((T_K(p^K)))|\mathbf{u}^K|^2 \,dx \le \frac{2||\mathbf{L}||^2 + \pi^2}{2\nu\pi^2}||\mathbf{f}||_{-1,2}^2,$$

and $\boldsymbol{\phi} = \nabla \left(p^K\right)^a$ where $a = 4l + 1$ for a natural number l in the same equation yields

$$\nu\left(\nabla \mathbf{u}^K, \nabla\nabla \left(p^K\right)^a\right) - \left(p^K, \Delta \left(p^K\right)^a\right) + \int_\Omega \alpha((T_K(p^K)))\mathbf{u}^K \cdot \nabla \left(p^K\right)^a \,\mathrm{d}x =$$
$$\langle \mathbf{f}_1 + \nabla f_2, \nabla \left(p^K\right)^a \rangle$$
$$-\nu\left(\Delta \mathbf{u}^K, \nabla \left(p^K\right)^a\right) + \left(\nabla p^K, \nabla \left(p^K\right)^a\right) + \int_\Omega \operatorname{div}\left(\alpha((T_K(p^K)))\mathbf{u}^K\right) \left(p^K\right)^a \,\mathrm{d}x =$$
$$\langle \mathbf{f}_1, \nabla \left(p^K\right)^a \rangle + \langle \nabla f_2, \nabla \left(p^K\right)^a \rangle$$
$$\nu\left(\Delta\mathrm{div}\ \mathbf{u}^K, \left(p^K\right)^a\right) + \left(\nabla p^K, \nabla \left(p^K\right)^a\right) + \int_\Omega \nabla\alpha((T_K(p^K))) \cdot \mathbf{u}^K \left(p^K\right)^a \,\mathrm{d}x =$$
$$-\langle \mathrm{div}\ \mathbf{f}_1, \nabla \left(p^K\right)^a \rangle + \langle \nabla f_2, \nabla \left(p^K\right)^a \rangle$$
$$\left(\nabla p^K, \nabla \left(p^K\right)^a\right) + \int_\Omega \alpha'((T_K(p^K)))\nabla p^K \xi_{[-K,K]}(p^K) \cdot \mathbf{u}^K \left(p^K\right)^a \,\mathrm{d}x =$$
$$\langle \nabla f_2, \nabla \left(p^K\right)^a \rangle$$
$$\left(\nabla p^K, \nabla \left(p^K\right)^a\right) + \int_\Omega \nabla A(p^K) \cdot \mathbf{u}^K \,\mathrm{d}x = \langle \nabla f_2, \nabla \left(p^K\right)^a \rangle$$
$$\left(\nabla p^K, \nabla \left(p^K\right)^a\right) - \left(A(p^K), \mathrm{div}\ \mathbf{u}^K\right) = \langle \nabla f_2, \nabla \left(p^K\right)^a \rangle$$

$$(4.3) \qquad \left(\nabla p^K, \nabla {p^K}^a\right) = \left(\nabla f_2, \nabla {p^K}^a\right).$$

Here we have used the Helmholtz decomposition of the right hand side $\mathbf{f}$, the solenoidality of $\mathbf{u}^K$, and partial integration together with the transcription of the term arisen from the testing the interaction term to $\left(\nabla A(p^K), \mathbf{u}^K\right)$ with $A(t) := \int_0^t \alpha'(T_K(s))\xi_{[-K,K]}(s)s^a \,ds$.

We further simplify (4.3)

$$\int_\Omega {p^K}^{a-1} \left|\nabla p^K\right|^2 \, dx = \int_\Omega {p^K}^{a-1} \nabla p^K \cdot \nabla f_2 \, dx \tag{4.4}$$

$$\left(\frac{2}{a+1}\right)^2 \int_\Omega \left|\nabla {p^K}^{\frac{a+1}{2}}\right|^2 \, dx \le \frac{2}{a+1} \int_\Omega \nabla {p^K}^{\frac{a+1}{2}} \cdot \nabla f_2 \, {p^K}^{\frac{a-1}{2}} \, dx \tag{4.5}$$

$$\frac{2}{a+1} \int_\Omega \left|\nabla {p^K}^{\frac{a+1}{2}}\right|^2 \, dx \le \frac{1}{a+1} \int_\Omega \left|\nabla {p^K}^{\frac{a+1}{2}}\right|^2 dx + \frac{a+1}{4} \int_\Omega \left|\nabla f_2 \, {p^K}^{\frac{a-1}{2}}\right|^2 dx \tag{4.6}$$

$$\frac{4}{(a+1)^2} \int_\Omega \left|\nabla {p^K}^{\frac{a+1}{2}}\right|^2 \, dx \le \int_\Omega |\nabla f_2|^2 \left|{p^K}^{\frac{a-1}{2}}\right|^2 \, dx. \tag{4.7}$$

Choosing $a := 1$ in (4.7) leads to the estimate $||p^K||_{W^{1,2}_{per}(\Omega)} \le C$ and allows to pass to the limit $K \to \infty$ in (2.4) for certain values of γ from (A1). In a similar way we get an estimate for $n = 2$

$$||p^K||_{BMO} \le C||\nabla f_2||_2. \tag{4.8}$$

To be able to take arbitrary finite γ we derive further with help of the embedding due to $(1.1)_5$ for $n \ge 3$

$$\left\|{p^K}^{\frac{a+1}{2}}\right\|^2_{\frac{2n}{n-2}} \le \frac{||\mathbf{L}||^2}{\pi^2} \left\|\nabla {p^K}^{\frac{a+1}{2}}\right\|^2_2. \tag{4.9}$$

Then

$$\|p^K\|^{a+1}_{\frac{n}{n-2}(a+1)} \le \frac{(a+1)^2 \frac{||\mathbf{L}||^2}{\pi^2}}{4} \int_\Omega |\nabla f_2|^2 \left|\nabla {p^K}^{\frac{a-1}{2}}\right|^2 \, dx \tag{4.10}$$

$$\|p^K\|_{\frac{n}{n-2}(a+1)} \le \left(\frac{(a+1)||\mathbf{L}||}{2\pi}\right)^{\frac{2}{a+1}} \left\||\nabla f_2|^2\right\|^{\frac{1}{a+1}}_{\frac{n(a+1)}{n(a+1)-(n-2)(a-1)}} \left\||p^K|^{a-1}\right\|^{\frac{1}{a+1}}_{\frac{n}{n-2}\frac{a+1}{a-1}} \tag{4.11}$$

This is amenable to the Young inequality with $q = \frac{a+1}{a-1}$ and $q^* = \frac{a+1}{2}$ leading to

$$\left(1 - \frac{a-1}{a+1}\right) \|p^K\|_{\frac{n}{n-2}(a+1)} \le \frac{2}{a+1} \left[\frac{\frac{||\mathbf{L}||}{\pi}(a+1)}{2}\right] \|\nabla f_2\|^{\frac{1}{4}}_{\frac{n(a+1)}{n+a-1}} \tag{4.12}$$

$$\|p^K\|_{\frac{n}{n-2}(a+1)} \le \frac{||\mathbf{L}||(a+1)}{2\pi} \|\nabla f_2\|^{\frac{1}{4}}_{n - \frac{n^2-2n}{n+a-1}}. \tag{4.13}$$

4.2. **Passage to the limits.** As was already mentioned the passage to the limit with $\epsilon \to 0+$ relies on the Lebesgue dominated convergence theorem whereas the limit $K \to \infty$ on its generalization, namely the Vitali convergence theorem. We state the theorem we use for a future reference.

Theorem 4.2 (Vitali convergence theorem). *Let (X,Σ,μ) be a measure space; let $p \geq 1$ and let $f_n : X \to \mathbb{R}$ be in the space $L^p(X,\Sigma,\mu;\mathbb{R})$ for each natural number $n \in \mathbb{N}$. Then f_n converges as $n \to \infty$ to another measurable function $f : X \to \mathbb{R}$ in L^p if and only if*

1. *f_n converges in measure to f;*
2. *the f_n are equiintegrable in the sense that $\forall \epsilon > 0 \quad \exists t \geq 0 \quad \forall n \in \mathbb{N}$ $\int_{\{x\in X | f_n(x) \geq t\}} |f_n(x)|\, d\mu(x) < \epsilon$*
3. *$\forall \epsilon > 0 \quad \exists E \subset X \quad \mu(E) < \infty \quad \forall n \in \mathbb{N} \int_{X\setminus E} |f_n(x)|^p \, d\mu(x) < \epsilon$.*

In the following we use the Banach-Alaoglu theorem many times without relabelling sequences.

4.2.1. *Passage to the limit with $\epsilon \to 0+$.* From $(\mathrm{EI}_{K,\epsilon})$ we obtain convergences

$$\mathbf{u}^{K,\epsilon} \xrightarrow[\left[W^{1,2}_{div,per}(\Omega)\right]^n]{\epsilon\to 0+} \mathbf{u}^K \tag{4.14}$$

$$\sqrt{\epsilon}\nabla p^{K,\epsilon} \xrightarrow[\left[W^{1,2}_{per}(\Omega)\right]^n]{\epsilon\to 0+} \mathbf{g}. \tag{4.15}$$

Now we may recover from $(2.3)_2$ the weak formulation of $(2.2)_2$

$$\lim_{\epsilon\to 0+} \left(\epsilon \nabla p^{K,\epsilon}, \nabla\psi\right) = \sqrt{\epsilon}\left(\mathbf{g}, \nabla\psi\right) = \left(\mathbf{u}^K, \nabla\psi\right)$$

estimating $\overline{\lim}_{\epsilon\to 0+} |\sqrt{\epsilon}\left(\mathbf{g},\nabla\psi\right)| \leq \overline{\lim}_{\epsilon\to 0+} \sqrt{\epsilon} \left\|\sqrt{\epsilon}\nabla p^{K,\epsilon}\right\|_2 ||\nabla\psi||_2 \leq$ $\overline{\lim}_{\epsilon\to 0+}\sqrt{\epsilon}\frac{||\mathbf{f}||^2_{-1,2}}{2\nu} = 0$ by the Cauchy-Schwarz inequality, $(\mathrm{EI}_{K,\epsilon})$ and (4.15).

From (4.2) we get the strong convergence of pressures $p^{K,\epsilon}$

$$p^{K,\epsilon} \xrightarrow[L^q_0(\Omega) \quad \forall 1\leq q<\frac{2n}{n-2}]{\epsilon\to 0+} p^K \quad (n \geq 3) \tag{4.16}$$

$$p^{K,\epsilon} \xrightarrow[L^2_0(\Omega)]{\epsilon\to 0+} p^K \quad (n = 2). \tag{4.17}$$

The convergences (4.14)–(4.17) already suffice to

$$\nu\left(\nabla\mathbf{u}^{K,\epsilon}, \nabla\boldsymbol{\phi}\right) \xrightarrow[(4.14)]{\epsilon\to 0+} \nu\left(\nabla\mathbf{u}^K, \nabla\boldsymbol{\phi}\right) \tag{4.18}$$

$$\left(p^{K,\epsilon}, \mathrm{div}\ \boldsymbol{\phi}\right) \xrightarrow[(4.16),(4.17)]{\epsilon\to 0+} \left(p^K, \mathrm{div}\ \boldsymbol{\phi}\right) \tag{4.19}$$

$$\left(\alpha\left(T_K\left(p^{K,\epsilon}\right)\right)\mathbf{u}^{K,\epsilon}, \boldsymbol{\phi}\right) \xrightarrow[(4.14),(4.16),(4.17)]{\epsilon\to 0+} \left(\alpha\left(T_K\left(p^K\right)\right)\mathbf{u}^K, \boldsymbol{\phi}\right) \tag{4.20}$$

implying the passage from (2.3) to (2.4). The last convergence in the non-linear term (4.20) still deserves an extra attention for $n \geq 3$. We have

$$\begin{aligned} &\left|\left(\alpha\left(T_K\left(p^{K,\epsilon}\right)\right)\mathbf{u}^{K,\epsilon}, \boldsymbol{\phi}\right) - \left(\alpha\left(T_K\left(p^K\right)\right)\mathbf{u}^K, \boldsymbol{\phi}\right)\right| \leq \\ &||\alpha||_{L^\infty([-K,K])}||\mathbf{u}^{K,\epsilon} - \mathbf{u}^K||_2||\boldsymbol{\phi}||_2 + \\ &||\alpha\left(T_K\left(p^{K,\epsilon}\right)\right) - \alpha\left(T_K\left(p^K\right)\right)||_{\frac{n}{2}}||\mathbf{u}^K||_{\frac{2n}{n-2}}||\boldsymbol{\phi}||_{\frac{2n}{n-2}} \end{aligned} \tag{4.21}$$

by the Hölder inequalities. The convergence to zero in the first term on the rhs of (4.21) follows from (4.14), and the Rellich(-Kondrashov) theorem, and the convergence in the second term is due to the fact that we may (by (4.16)) extract a subsequence of $p^{K,\epsilon}$ converging almost everywhere in Ω and use the Lebesgue dominated convergence theorem with a dominating function — $2^{\frac{n}{2}} \sup_{s\in[-K,K]}^{\frac{n}{2}} \alpha(s)\xi(x)$. The case $n=2$ is similar as a dominating function can be chosen as $2^p \sup_{s\in[-K,K]}^{p} \alpha(s)\xi(x)$ for a $p>1$.

4.2.2. *Passage to the limit with $K\to\infty$.* This case resembles the previous one except for (4.20). The energy inequality (EI_K) gives

$$\mathbf{u}^K \xrightarrow[\left[W^{1,2}_{div,per}(\Omega)\right]^n,(\mathrm{EI}_K)]{K\to\infty} \mathbf{u}. \tag{4.22}$$

By (4.13), resp. (4.8) we obtain

$$p^K \xrightarrow[L^q_0(\Omega)\quad \forall 1\le q<\infty]{K\to\infty} p. \tag{4.23}$$

The weak convergence in (4.23) is in fact a strong one since by the boundedness of the pressure gradient (4.7) we may interpolate

$$||p^K-p||_q \le ||p^K-p||^{\delta}_{\frac{2n}{n-2}-\epsilon}||p^K-p||^{1-\delta}_{q'} = o\left(\frac{1}{K}\right) \tag{4.24}$$

for every $\frac{2n}{n-2} < q' < \infty$ and a certain $\frac{2n}{n-2} < q < q'$, $0<\epsilon<<1$ and $\delta\in(0,1)$. Indeed, the term $||p^K-p||^{1-\delta}_{q'}$ is bounded by (4.23) and the term $||p^K-p||^{\delta}_{\frac{2n}{n-2}-\epsilon}$ converges to zero for $K\to\infty$, and $n\ge 3$ by (4.7) with $a=1$ and Sobolev compact embedding theorem $W^{1,2}_{per}(\Omega)\subset\subset L^{\frac{2n}{n-2}-\epsilon}(\Omega)$. The case $n=2$ is handled by interpolation between $L^2_0(\Omega)$ and $L^q_0(\Omega)$ for any $q>2$.

Therefore

$$p^K \xrightarrow[L^q_0(\Omega)\quad \forall 1\le q<\infty]{K\to\infty} p. \tag{4.25}$$

Once again

$$\nu\left(\nabla\mathbf{u}^K,\nabla\boldsymbol{\phi}\right) \xrightarrow[(4.22)]{K\to\infty} \nu\left(\nabla\mathbf{u},\nabla\boldsymbol{\phi}\right) \tag{4.26}$$

$$\left(p^K,\mathrm{div}\ \boldsymbol{\phi}\right) \xrightarrow[(4.23)]{K\to\infty} \left(p,\mathrm{div}\ \boldsymbol{\phi}\right) \tag{4.27}$$

$$\left(\alpha\left(T_K\left(p^K\right)\right)\mathbf{u}^K,\boldsymbol{\phi}\right) \xrightarrow[(4.22),(4.25)]{K\to\infty} \left(\alpha\left(p\right)\mathbf{u},\boldsymbol{\phi}\right) \tag{4.28}$$

the last convergence needs an additional explanation. We observe that the dominating function now blows up and an estimate as in (4.21)

$$\left|\left(\alpha\left(T_K\left(p^K\right)\right)\mathbf{u}^K,\boldsymbol{\phi}\right)-\left(\alpha\left(p\right)\mathbf{u},\boldsymbol{\phi}\right)\right| \le \left\|\alpha\left(T_K\left(p^K\right)\right)\right\|_{\frac{n}{2}}||\mathbf{u}^K-\mathbf{u}||_2||\boldsymbol{\phi}||_\delta + ||\alpha\left(T_K\left(p^K\right)\right)-\alpha\left(p\right)||_{\frac{n}{2}}||\mathbf{u}^K||_{\frac{2n}{n-2}}||\boldsymbol{\phi}||_{\frac{2n}{n-2}} \tag{4.29}$$

holds with $\delta := \frac{n[2n-(n-2)\zeta]}{n^2-2n-\zeta(n-2)^2}$ for any small $\zeta>0$. We verify the hypotheses of the Vitali convergence theorem (4.2). Naturally, we set $X=\Omega$, Σ–the σ–algebra of Lebesgue measurable sets in Ω, $\mu=\lambda_n$ the n–dimensional

Lebesgue measure, $f_K := \alpha\left(T_K(p^K)\right)$, and $p := \frac{n}{2}$. The third hypothesis is automatically satisfied as Ω is bounded. For the first one, we note that p^K converges in measure in Ω thanks to (4.25). This means that

$$\text{(pcm)} \qquad \forall \epsilon > 0 \ \lim_{K\to\infty} \lambda_n \left(\left\{x \in \Omega \mid \left|p(x) - p^K(x)\right| \geq \epsilon\right\}\right) = 0$$

by the definition. For the corresponding difference between $\alpha\left(T_K\left(p^K\right)\right) - \alpha(p)$ we dissect the measure $\lambda_n\left(\left\{x \in \Omega \mid \left|\alpha(T_k(p^K(x)) - \alpha(p(x))\right| \geq \epsilon\right\}\right)$ by a level t of the function $\alpha\left(T_K\left(p^K\right)\right)$ to obtain two contributions

$$\mathcal{M}^1_K := \lambda_n\left(\left\{x \in \Omega \mid \alpha(T_K(p^K(x)) \leq t \,\&\, \left|\alpha(T_K(p^K(x)) - \alpha(p(x))\right| \geq \epsilon\right\}\right)$$
$$\mathcal{M}^2_K := \lambda_n\left(\left\{x \in \Omega \mid \alpha(T_K(p^K(x)) > t \,\&\, \left|\alpha(T_K(p^K(x)) - \alpha(p(x))\right| \geq \epsilon\right\}\right).$$

The first contribution $\mathcal{M}^1_K$ is easily controlled by a local Lipschitzian constant A_t [2] of the given function $\alpha(s)$ on the preimage $\alpha^{-1}(0,t]$ and tends to zero by the convergence in measure of p^K

$$\mathcal{M}^1_K \leq \lambda_n\left(\left\{x \in \Omega \mid \alpha(T_K(p^K(x)) \leq t \,\&\, A_t\left|p^K(x) - p(x)\right| \geq \epsilon\right\}\right) \leq \delta,$$

where $\delta > 0$ is any small number used in the definition of the limit of the sequence of measures of sets in (pcm).

The second contribution $\mathcal{M}^2_K$ also vanishes in the limit as we show even more (for $n \geq 3$) in the next paragraph.

It remains to verify the equiintegrability of $\alpha\left(T_K\left(p^K\right)\right)$ in $L^{\frac{n}{2}}(\Omega)$ (the second hypothesis from the (4.2)):

$$\text{(4.30)} \quad \forall \epsilon > 0 \quad \exists t \geq 0 \quad \forall K \in \mathbb{N} \qquad \int_{\{x\in\Omega|\alpha(T_K(p^K(x)))\geq t\}} \left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|^{\frac{n}{2}} dx < \epsilon.$$

We fix γ in (A1), the space dimension n and utilize (4.25) with a certain $q = r$. A direct estimate thanks to the Hölder (Chebyshev) inequality and (4.13) implies a uniform bound (in K)

$$\int_{\{x|\alpha(T_K(p^K(x)))\geq t\}} \left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|^{\frac{n}{2}} \,\mathrm{d}x \leq$$
$$\leq C\, 2^{\frac{n-2}{2}} \left\{ \frac{C\left(\frac{r}{\gamma}, \Omega\right) + \left(\frac{||\mathbf{L}||}{\pi}\right)^r ||\nabla f_2||_n^{\frac{r}{4}} \left(\frac{n-2}{nr^{-1}}\right)^r}{t^{\frac{r}{\gamma}}} + C(\Omega)\times \right.$$
$$\left. \times \left[\frac{C\left(\frac{r}{\gamma}, \Omega\right) + \left(\frac{||\mathbf{L}||}{\pi}\right)^r ||\nabla f_2||_n^{\frac{r}{4}} \left(\frac{n-2}{nr^{-1}}\right)^r}{t^{\frac{r}{\gamma}}}\right]^{\frac{2r-n\gamma}{2r}} \pi^{-\frac{nr\gamma}{2}} ||\nabla f_2||_n^{\frac{n\gamma}{8}} \left(\frac{n-2}{nr^{-1}}\right)^{\frac{n\gamma}{2}} \right\} \leq$$
$$\text{(4.31)} \qquad \leq C(n, r, \gamma, \Omega, f_2)\left[t^{-\frac{r}{\gamma}} + t^{-\frac{2r-n\gamma}{\gamma}}\right]$$

where $C(n, r, \gamma, \Omega, f_2)$ is a positive function independent of K if $r > \frac{n\gamma}{2}$ is chosen. In particular, we recover that the sequence $\left\{\alpha\left(T_K\left(p^K\right)\right)\right\}_{K=1}^{\infty}$ has

[2] In fact α uniformly continuous is sufficient here.

uniformly absolutely continuous integrals

$$\mathcal{M}_K^2 \leq \lambda_n\left(\{x\in\Omega \mid \alpha\left(T_K\left(p^K\right)\right)(x)\geq t\}\right)\leq$$
$$\int_{\{x\in\Omega|\alpha(T_K(p^K))(x)\geq t\}}\left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|\,\mathrm{d}x\leq$$
$$\leq\int_{\{x\in\Omega|\alpha(T_K(p^K(x)))\geq t\}}\left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|^{\frac{n}{2}}\,\mathrm{d}x\leq$$
$$\leq 2C(n,n\gamma,\gamma,\Omega,f_2)t^{-n}<\epsilon$$

for $r=n\gamma, t\geq\max\left(1,\sqrt[n]{2C\epsilon^{-1}}\right)$.

The estimate (4.31) helps to control both terms on the rhs of (4.29) together with the strong convergence of velocities $\mathbf{u}^K$ in the space $L^{\frac{2n}{n-2}-\epsilon}(\Omega)\subset L^2(\Omega)$ ((4.22) and the compact Sobolev embedding, $0<\epsilon<<1$). A density consideration for the function $\boldsymbol{\phi}$ finishes the proof.

The case $n=2$ relies again on the Vitali convergence theorem. The equiintegrability in the case of the so-called Barus' law (i. e. an exponential dependence $\alpha(t)=C_1e^{C_2t}$, with $C_1,C_2>0$ instead (A2)) is a direct consequence of the John-Nirenberg inequality[4]

$$\lambda_2\left\{x\in\Omega \mid \left|p^K(x)-\frac{1}{L_1L_2}\int_\Omega p^K(x)\right|>t\right\}\leq$$
$$\sqrt{2}\sqrt{\sqrt{2}+1}L_1L_2e^{-\frac{t(3-2\sqrt{2})\ln\frac{\sqrt{2}+1}{2}}{32||p^K||_{BMO}}}$$

and uniform boundedness in the space of bounded mean variation ($\sup_{K\in\mathbb{N}}||p^K||_{BMO}=:D\leq\infty$). Namely, if we want to estimate $\int_{\{x\in\Omega|\alpha(T_K(p^K))(x)\geq t\}}\left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|\,\mathrm{d}x$ for Barus' law and $n=2$, we observe that

$$\alpha\left(p^K(x)\right)>u\iff p^K(x)>C_2^{-1}\ln\frac{u}{C_1}.$$

That is why the second hypothesis of the Vitali theorem holds also in this case:

$$\int_{\{x\in\int_\Omega|\alpha(T_K(p^K(x)))\geq t\}}\left|\alpha\left(T_K\left(p^K(x)\right)\right)\right|\,\mathrm{d}x\leq$$
$$\leq\sqrt{2}\sqrt{\sqrt{2}+1}L_1L_2e^{-\frac{(3-2\sqrt{2})\ln\frac{t}{C_1}\ln\frac{\sqrt{2}+1}{2}}{32C_2D}}\leq\epsilon.$$

Moreover, the Sobolev embedding into the exponential Orlicz class corresponding to the N-function $e^{t^2}-1$ and strong continuity of the Nemytskii operator as a special[3] case of the composition operator from this class into $L^p(\Omega)$ with $p>1$ (as cited in Lemma 1.3 of [1]) enables to deal with the growth condition (A2) as well.

The energy inequality (EI) is valid because of the weak lower semicontinuity of the L^2-seminorm in the space $\left[W^{1,2}_{div,per}(\Omega)\right]^n$ in and a slightly modified argument in the Vitali theorem with the summability exponent $p>\frac{n}{2}$ yielding

$$\alpha\left(T_K(p^K)\right)\to\alpha(p)\qquad\text{in } L^p(\Omega)$$

[3] i. e. spatially homogeneous

and from (4.22)

$$|\mathbf{u}^K|^2 \to \mathbf{u}^K \qquad \text{in } L^{\frac{p}{p-1}}(\Omega),$$

so that

$$\alpha\left(T_K(p^K)\right)|\mathbf{u}^K|^2 \to \alpha(p)\mathbf{u}^K \qquad \text{in } L^1(\Omega).$$

5. Conclusion

In this paper we proved a first result concerning the existence of a weak solution to the Brinkman-like system governing fluid flow in porous media. It represents just the first step in a hierarchy of models describing this phenomenon with increasing fidelity compared to real life situations. In the forthcoming papers, its regularity and a maximum principle will be treated. The essential role will still play that we may neglect the convective term for creeping flows.

We have seen that we have needed to confine ourselves to the more regular right hand side $\mathbf{f}$ in order to get a strong convergence. The general "natural" case with $\mathbf{f}$ in a negative Sobolev space and inflow-outflow boundary conditions remains open. However, since under suitable assumption on the drag coefficient α and applied force $\mathbf{f}$ we can essentially bound the pressure function, we can treat superexponential and exponential classes of growth behaviour of the function α like the widely used Barus' law.

We have treated the case of periodic boundary conditions only. In general "natural" boundary conditions involving the stress tensor of the fluid are relevant. In such cases the well-posedness of the problem of this (and extended problems capturing fast changes of the flow correctly) remain open.

A study of the system (1.1) signifies also a preparatory step made before taking the limit $\tau \to 0+$. This can imply the existence of a weak solution to an initial value problem for the evolutionary version of the system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p + \alpha(p)\mathbf{u} &= \mathbf{f} \quad \text{in } Q_T \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } Q_T \\ \mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega \\ \mathbf{u}, p &\quad \Omega - \text{periodic} \\ \int_\Omega p(x)\,dx &= 0 \quad \forall t \in [0,T) \end{aligned} \tag{5.1}$$

The first results in this direction were provided by [8]. Those problems have applications in the problems of carbone dioxide sequestration and enhanced oil recovery as stressed by [6].

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