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NON-EUCLIDEAN CRYSTAL GEOMETRY

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To Honour of János BOLYAI on the 220th Anniversary of His Birth.

ABSTRACT. The discovery of hyperbolic geometry $\mathbf{H}^3$ by János BOLYAI and Nikolay Ivanovich LOBACHEVSKY open also new directions in material sciences. Besides of 3-spaces of constant curvature: $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, other five homogeneous 3-spaces: $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$, $\mathbf{Nil}$, $\widetilde{\mathbf{SL_2R}}$, $\mathbf{Sol}$ (so-called Thurston geometries) come into considerations. In analogies of classical crystallography, ball-packing models deserve investigations and yield interesting new results for comparing with the real crystals, extremal arrangements, and open problems.

1. INTRODUCTION

W. Thurston's results are essential for understanding the geometric structure of our world, where the eight so-called Thurston geometries play the leading role. The importance of these geometries is emphasized by Thurston's famous theorem as follows in Theorem 1.1.

Let $(X;\mathbf{G})$ be a 3-dimensional homogeneous geometry, where X is a simply connected Riemannian space with a maximal group $\mathbf{G}$ of isometries, acting transitively on X with compact point stabilizers. $\mathbf{G}$ is maximal means that no proper extension of $\mathbf{G}$ can act on the Riemannian space X in the same way. We recall the

Theorem 1.1 (Thurston, [22, 33])**.** *Any* 3*-dimensional homogeneous geometry* $(X;\mathbf{G})$ *that admits a compact quotient is equivalent (equivariant) to one of the geometries* $(X;\mathbf{G})$. $\mathbf{G}=Isom(X)$ *where the space* X *is one of* $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$, $\mathbf{H}^2\times\mathbf{R}$, $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{Nil}$, $\widetilde{\mathbf{SL_2R}}$ *or* $\mathbf{Sol}$.

Therefore, there are eight so-called Thurston geometries, described in [22, 33]. Among them $\mathbf{E}^3$, $\mathbf{S}^3$ and $\mathbf{H}^3$ are the classical spaces of constant zero, positive and negative curvature, respectively. Further geometries $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$ denote the direct product geometries where $\mathbf{S}^2$ is the spherical and $\mathbf{H}^2$ is the hyperbolic base plane and the real line $\mathbf{R}$ is with usual metric. Then $\widetilde{\mathbf{SL_2R}}$ and $\mathbf{Nil}$ are obtained as twisted products of $\mathbf{R}$ with $\mathbf{H}^2$ and $\mathbf{E}^2$, respectively; and finally $\mathbf{Sol}$ geometry is a twisted product of the Minkowski plane $\mathbf{M}^2$ as fibre, with $\mathbf{R}$ as base. In each of them there exists an infinitesimal (positive definite) Riemannian metric that is invariant under certain translations, guaranteeing homogeneity at every point. These translations in general commute only in $\mathbf{E}^3$, but a discrete

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(discontinuous) translation group, taken as a lattice, can be defined with compact fundamental domain in analogy to the Euclidean case, but with some different properties. The additional symmetries can define crystallographic groups, giving nice tilings, packings, material structures, etc.

We mention here only the sphere (ball) packing problems. In addition to pure mathematical curiosity, the study of sphere packings the *generalized Kepler problem* is important because it is possible that under certain conditions (e.g. strong magnetic field) materials cannot be realized in the usual Euclidean space but in one of the other Thurston geometries. The structures of substances formed under these conditions may differ from the Euclidean case and can follow, for example, the geometry of non-constant curvature spaces, and in these new geometries their atoms can be modelled by $\mathbf{H}^2\times\mathbf{R}$, $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{Nil}$, $\widetilde{\mathbf{SL_2R}}$ or $\mathbf{Sol}$ spheres. For example, in $\mathbf{Nil}$ geometry we can define lattices and corresponding lattice-like ball packings where we found geodesic ball packings with kissing number 14 that is denser than the densest Euclidean case (see [11], [23]). (The density is ≈ 0.78085).

A unified approach to Thurston geometries enabling the investigations in this direction were made possible by the paper of E. Molnár [7] where he showed that the Thurston geometries can be uniformly modelled in the projective 3-space $\mathcal{P}^3$, or in the projective 3-sphere $\mathcal{PS}^3$. This projective spherical model is based on linear algebra over the real vector space $\mathbf{V}^4$ (for points) and its dual $\boldsymbol{V}_4$ (for planes), up to a positive real factor, so that the proper dimension is indeed three. A plane $\to$ point polarity or scalar product (by specified signature) induces the invariant metric in a unified way. In our work we will use these projective models of Thurston geometries (Table 1).

The constant curvature geometries $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$ have been extensively studied from the point of view of elementary geometry, differential geometry and topology. In this article we focus on results obtained in the other five Thurston geometries $\mathbf{H}^2\times\mathbf{R}$, $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{Nil}$, $\widetilde{\mathbf{SL_2R}}$, $\mathbf{Sol}$. These spaces have been investigated from the perspective of differential geometry and topology but few results are stated in connection with their internal structure in the classical sense. Hence, in this survey we focus on non-constant curvature Thurston geometries and we emphasize some surprising facts.

We review the concepts of sphere (ball) packings and their densities and the corresponding results so far.

Furthermore, we emphasize the results related to the projective models of the considered geometries. In our opinion, these models are suitable for the elementary examination and visualization of the above geometries as well.

Remark 1.2. There is another way of defining distance using the concept of so-called *translational distance*. We introduced this concept in paper [13], but in this survey we summarize the results related to the concept of geodesic distance. Note that translation distance and geodesic distance are the same in the Euclidean geometry $\mathbf{E}^3$, Bolyai–Lobachevsky hyperbolic geometry $\mathbf{H}^3$, spherical $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ spaces, but give different values in $\mathbf{Nil}$, $\mathbf{Sol}$ and $\widetilde{\mathbf{SL_2R}}$ geometries (see [20, 30]).

As the reader will see, the above results and their visualizations will open a new window towards other (geometric) worlds [35].

Table 1

The eight Thurston geometries modelled in $\mathcal{PS}^3$ by a polarity or scalar product and its isometry group.

Space $\mathbf{X}$	Signature of polarity $\Pi(\star)$ or scalar product $\langle\ ,\ \rangle$ in $\boldsymbol{V}_4$	Domain of proper points of $\mathbf{X}$ in $\mathcal{PS}^3$ $(\mathbf{V}^4(\mathbf{R}),\ \boldsymbol{V}_4)$	The group $G = \mathrm{Isom}\,\mathbf{X}$ as a special collineation group of $\mathcal{PS}^3$
$\mathbf{S}^3$	$(+ + ++)$	$\mathcal{PS}^3$	Coll $\mathcal{PS}^3$ preserving $\Pi(\star)$
$\mathbf{H}^3$	$(- + ++)$	$\{(\mathbf{x}) \in \mathcal{P}^3\ :\ \langle \mathbf{x}, \mathbf{x}\rangle < 0\}$	Coll $\mathcal{P}^3$ preserving $\Pi(\star)$
$\widetilde{\mathbf{SL_2R}}$	$(- - ++)$ with skew line fibering	Universal covering of $\mathcal{H} :=$ $:= \{[\mathbf{x}] \in \mathcal{PS}^3\ :\ \langle \mathbf{x}, \mathbf{x}\rangle < 0\}$ by fibering transformations	Coll $\mathcal{PS}^3$ preserving $\Pi(\star)$ and fibres with 4 parameters.
$\mathbf{E}^3$	$(0 + ++)$	$\mathcal{A}^3 = \mathcal{P}^3 \setminus \{\omega^\infty\}$ where $\omega^\infty := (\boldsymbol{b}^0),\ \boldsymbol{b}^0_\star = \mathbf{0}$	Coll $\mathcal{P}^3$ preserving $\Pi(\star)$, generated by plane reflections
$\mathbf{S}^2\times\mathbf{R}$	$(0 + ++)$ with O-line bundle fibering	$\mathcal{A}^3 \setminus \{O\}$ O is a fixed origin	G is generated by plane reflections and sphere inversions, leaving invariant the O-concentric 2-spheres of $\Pi(\star)$
$\mathbf{H}^2\times\mathbf{R}$	$(0 - ++)$ with O-line bundle fibering	$\mathcal{C}^+ = \{X \in \mathcal{A}^3\ :$ $\langle\overrightarrow{OX}, \overrightarrow{OX}\rangle < 0$, half cone$\}$ by fibering	G is generated by plane reflections and hyperboloid inversions, leaving invariant the O-concentric half-hyperboloids in the half-cone $\mathcal{C}^+$ by $\Pi(\star)$
$\mathbf{Sol}$	$(0 - ++)$ and parallel plane fibering with an ideal plane ϕ	$\mathcal{A}^3 = \mathcal{P}^3 \setminus \phi$	Coll. of $\mathcal{A}^3$ preserving $\Pi(*)$ and the fibering with 3 parameters
$\mathbf{Nil}$	Null-polarity $\Pi(\star)$ with parallel line bundle fibering F with its polar ideal plane ϕ	$\mathcal{A}^3 = \mathcal{P}^3 \setminus \phi$	Coll. of $\mathcal{A}^3$ preserving $\Pi(\star)$ with 4 parameters

2. BALL PACKINGS IN THURSTON GEOMETRIES

2.1. **Geodesic ball packings in spaces of constant curvature.** Let X denote a space of constant curvature, either the n-dimensional sphere $\mathbf{S}^n$, Euclidean space $\mathbf{E}^n$, or hyperbolic space $\mathbf{H}^n$ with $n \geq 2$. An important question of discrete geometry is to find the highest possible packing density in X by congruent non-overlapping balls of a given radius [4].

FIGURE 1. No pictures of János BOLYAI exist. Recently,the opinion that one of the reliefs of the Palace of Culture at Marosvásárhely (Târgu Mureş, Romania) portrays him (also with his name, near father's relief) has gained acceptance. Moreover, there is a striking similarity between the relief and the portrait of György Klapka, a general of the Hungarian revolutionary army of 1848-49. It is known that János resembled György Klapka. This is the most accepted picture of János BOLYAI, published by Tibor Weszely, Természet Világa 2018. jun. 262-264.

Euclidean cases are the best explored. One major recent development has been the settling of the long-standing Kepler conjecture, part of Hilbert's 18th problem, by Thomas Hales at the turn of the 21st century. Hales' computer-assisted proof was largely based on a program set forth by L. Fejes Tóth in the 1950's.

In n-dimensional hyperbolic geometry several new questions occur concerning packing and covering problems, e.g., in $\mathbf{H}^n$ there are 3 kinds of "generalized balls (spheres)": the *usual balls* (spheres), *horoballs* (horospheres) and *hyperballs* (hyperspheres [31], [32]). Moreover, the definition of packing density is crucial in hyperbolic spaces as shown by Böröczky [1]. For standard examples also see [4], [5]. The most widely accepted notion of packing density considers the local densities of balls with respect to their Dirichlet–Voronoi cells (cf. [1]). In order to consider ball packings in $\overline{\mathbf{H}}^n$, we use an extended notion of such local density. In space X^n let $d_n(r)$ be the density of $n+1$ mutually touching spheres or horospheres of radius r (in case of horosphere $r = \infty$) with respect to the simplex spanned by their centres. L. Fejes Tóth and H. S. M. Coxeter conjectured that the packing density of balls of radius r in X^n cannot exceed $d_n(r)$. This conjecture has been proved by C. A. Rogers for the Euclidean space $\mathbf{E}^n$. The 2-dimensional spherical case was settled by L. Fejes Tóth in [5].

In [1] K. Böröczky claimed the following theorem for *ball and horoball* packings for any dimension $2 \leq n \in \mathbf{N}$:

In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. In spherical space suppose that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet–Voronoi cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centres.

This density is ≈ 0.85328 in $\mathbf{H}^3$ which is not realized by packings with equal balls. However, it is attained by the horoball packing (case $r = \infty$) of $\overline{\mathbf{H}}^3$ where the ideal centres of horoballs lie on the absolute figure of $\overline{\mathbf{H}}^3$. This corresponds to packing an ideal regular tetrahedron tiling given by the Coxeter–Schläfli symbol $\{3,3,6\}$. But $\{3,4,6\}$ with cubes of ideal vertices leads to the same density of horoball packing. Ball packings of hyperbolic n-space and of other Thurston geometries are extensively discussed in the literature see e.g. [4, 12, 31, 32], where the reader finds further references as well.

In this survey, we do not deal in details with the ball (sphere) packings and coverings of spaces of constant curvature, so now we only mention that the questions regarding horosphere and hypersphere packings and coverings are not yet settled. Moreover, the famous football manifold [12] provides the densest known (classical) ball packing configuration of density ≈ 0.77147, with parameters $(u,v,w) = (5;3;5)$ in $\mathbf{H}^3$ (see Fig. 2). An infinite series of hyperbolic space groups is described in [12] and [9] with possible packings and subgroups with manifold structures. New interesting problems have also arosen, which are related to the Busemann functions. The interested reader can read about the results of these in the papers [2, 6] and the references therein.

2.2. **Geodesic ball packings in Thurston geometries of non-constant curvature.** Definitions of ball (sphere) packing and covering densities are already critical in hyperbolic geometry, therefore in order to introduce this concept to Thurston geometries of non-constant curvature we use the discrete isometry groups of them. First, we have summarized the basic definitions and notions (see [24]).

Let X be one of the five Thurston geometries of non-constant curvature

$$\mathbf{S}^2\!\times\!\mathbf{R},\ \mathbf{H}^2\!\times\!\mathbf{R},\ \widetilde{\mathbf{SL_2R}},\ \mathbf{Nil},\ \mathbf{Sol},$$

where the geodesic curves are generally defined as having locally minimal arc length between any two of their points (sufficiently close to each other). The system of equations for the parametrized geodesic curves $\gamma(\tau)$ in our model can be determined by the general theory of Riemannian geometry. *Then a geodesic sphere and ball can usually be defined. We consider only geodesic ball packings which are generated by discrete groups of isometries of X and the density of the packing is related to its Dirichlet–Voronoi cells.*

In the following, let Γ be a fixed group of isometries of X. Denote by $d(P_1, P_2)$ the geodesic distance of two points P_1, P_2.

Definition 2.1. We say that the point set

$$\mathcal{D}(K) = \{P \in X \, : \, d(K,P) \leq d(K^{\mathbf{g}}, P) \text{ for all } \mathbf{g} \in \Gamma\}$$

is the Dirichlet–Voronoi cell ($D-V$ cell) of Γ around the kernel point $K \in X$.

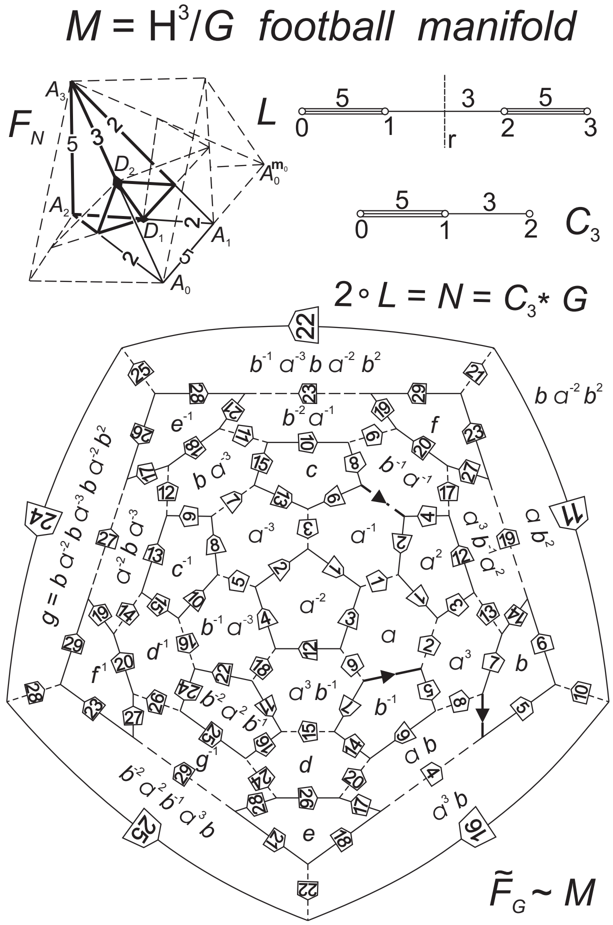

FIGURE 2. Football manifold with its fundamental domain with paired faces, described in the Beltrami-Cayley-Klein model of $\mathbf{H}^3$. A probable model for fullerenes [16].

2.2.1. *Simply transitive ball packings.* Let Γ be a fixed group of isometries in the space X. *Our goal is* to find a point $K \in X$ and the orbit K^Γ for Γ such that $\Gamma_K = \mathbf{I}$ and the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal. In this case the ball packing $\mathcal{B}^\Gamma(K)$ is said to be *optimal.*

We have to determine the maximal radius $\rho(K)$ of the balls, and the maximal density $\delta(K)$. The space groups considered could have free parameters. So we have to find the densest ball packing for fixed parameters $p(\Gamma)$, and then we have to vary $p(\Gamma)$ to get the optimal ball packing

$$\delta(\Gamma) = \max_{K,\ p(\Gamma)} (\delta(K)). \tag{2.1}$$

We look for the optimal kernel point in a 3-dimensional region, contained in a fundamental domain of Γ.

2.2.2. *Multiply transitive ball packings.* Similarly to the simply transitive case we must find a kernel point $K \in X$ and the orbit K^Γ of Γ such that the density $\delta(K)$ of the

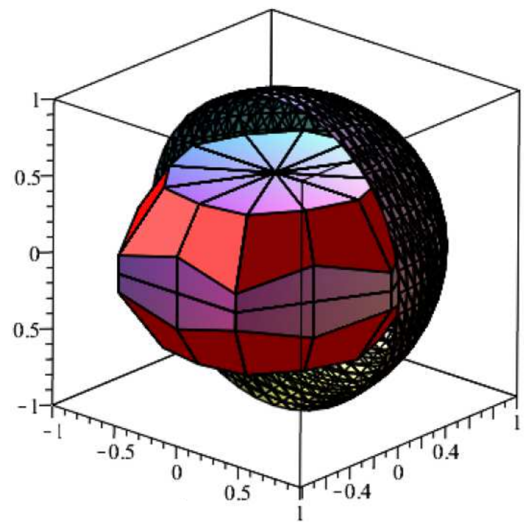

FIGURE 3. The fundamental domain of our analogous tube manifold $Cw(6)$ in $\mathbf{H}^3$, from among the tube manifolds $Cw(2p)$ with face pairing of p-rotation symmetry [16].

corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal, but here $\Gamma_K \neq \mathbf{I}$. Such a ball packing $\mathcal{B}^\Gamma(K)$ is also called *optimal*. In this multiply transitive case we look for the optimal kernel point K in possible 0-, 1-, or 2-dimensional regions $\mathcal{L}$, respectively.

3. GEODESIC BALL PACKINGS IN $\mathbf{Nil}$

W. Heisenberg's famous real matrix group (see e.g. [18]) provides a non-commutative translation group of an affine 3-space. $\mathbf{Nil}$ geometry can be derived from this matrix group.

In [8, 23] we investigated the geodesic balls of $\mathbf{Nil}$ and computed their volume, introduced the notion of the $\mathbf{Nil}$ lattice, $\mathbf{Nil}$ parallelepiped and the density of the lattice-like ball packing. Moreover, we have determined the densest lattice-like geodesic ball packing by a family of $\mathbf{Nil}$ lattices. The density of this packing is $\approx \mathbf{0.78085}$, which may be surprising enough in comparison with the 3-dimensional analogous Euclidean result $\frac{\pi}{\sqrt{18}} \approx 0.74048$. The kissing number of every ball in this packing is 14 (Fig. 4, 5). *We conjecture that in* $\mathbf{Nil}$ *space the densest geodesic ball packing belongs to the above ball arrangement.* The symmetry group of this packing has also been described in [10, 11].

4. GEODESIC BALL PACKINGS IN $\mathbf{H}^2 \times \mathbf{R}$

This space is derived from the direct product of the hyperbolic plane $\mathbf{H}^2$ and the real line $\mathbf{R}$. In [29] we determined the geodesic balls of $\mathbf{H}^2 \times \mathbf{R}$ and computed their volume, defined the notion of the geodesic ball packing and its density. Moreover, we have developed a procedure [29] to determine the density of the simply or multiply transitive geodesic ball packings for generalized Coxeter space groups of $\mathbf{H}^2 \times \mathbf{R}$ and applied this algorithm to them. For the above space groups the Dirichlet–Voronoi cells are "prisms" in the $\mathbf{H}^2 \times \mathbf{R}$ sense. The optimal packing density of the generalized Coxeter space groups is $\approx \mathbf{0.60726}$. We are sure, that in this space there are denser ball packings. The problem is open yet.

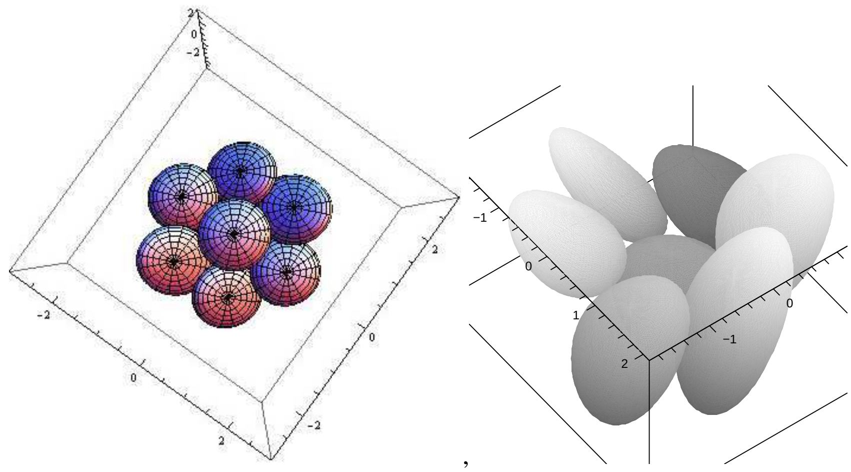

FIGURE 4. The densest geodesic lattice-like geodesic ball packing in **Nil** space.

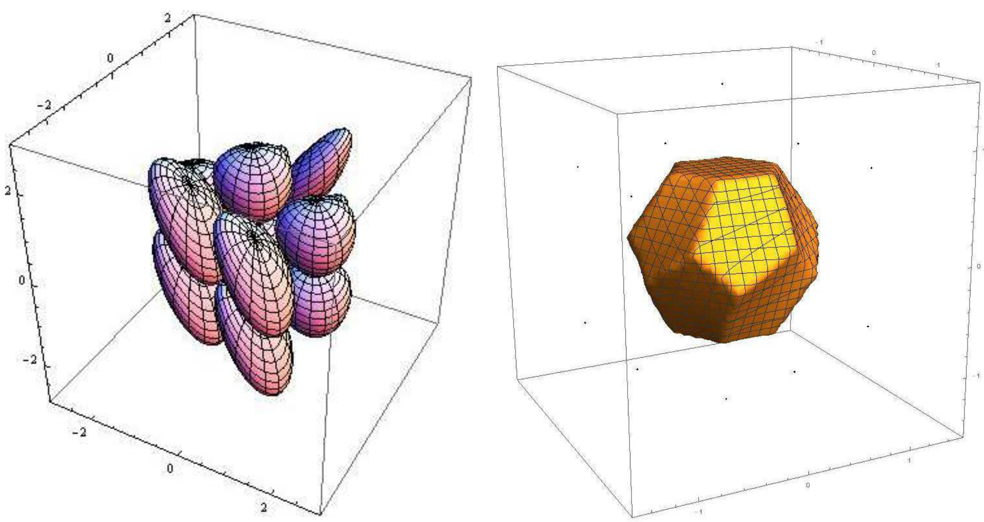

FIGURE 5. The densest geodesic lattice-like geodesic ball packing in **Nil** space and the corresponding Dirichlet–Voronoi cell.

5. GEODESIC BALL (SPHERE) PACKINGS IN $\widetilde{\mathbf{SL_2R}}$ SPACE

In [25] we investigated the regular prisms and prism tilings in $\widetilde{\mathbf{SL_2R}}$ and in [15] we considered the problem of geodesic ball packings related to tilings and their symmetry groups $\mathbf{pq2_1}$. Moreover, we computed the volumes of prisms and defined the notion of geodesic ball packing and its density. In [15] we developed a procedure to determine the densities of the densest geodesic ball packings for the tilings considered, more precisely, for their generating groups $\mathbf{pq2_1}$ (for integer rotational parameters p, q; $3 \leq p,\ \frac{2p}{p-2} < q$). We looked for those parameters p and q above, where the packing density as largest as possible. In these cases our record is 0.5674 for $(p,q) = (8,10)$. In [26] we studied the non-periodic geodesic ball packings related to the prism tilings and of the cases examined, the highest density that occurs is ≈ 0.6266.

In [20] we considered tilings $\mathcal{T}(p,(q,k),(o,\ell))$ for suitable integer positive parameters p,q,k,o,ℓ. Every tiling $\mathcal{T}$ is generated by discrete isometry group $\mathbf{p}\mathbf{q}_k\mathbf{o}_\ell$ for $k=1, o=2$, $\ell=1$. That means this group is generated by a $p-$rotation $\mathbf{p}$ about the central fibre, then by $\mathbf{q}_k$ screw with $q-$rotation and $\frac{k}{q}$ translation, then by an $\mathbf{o}_\ell$ screw with $o-$rotation and $\frac{\ell}{o}$ translation, just by Euclidean analogy but exact projective computations. We computed the maximal density of the ball packings induced by the $\mathbf{p}\mathbf{q}_k\mathbf{o}_\ell$ group action for any parameters. In the next Table 2 we have summarized some numerical results with the top density ≈ 0.787758. The table contain the optimal radius ρ_{opt}, the volume of the ball $B(\rho_{\text{opt}})$, the volume of the prism $\mathcal{P}_p$, and the packing density $\delta(\rho_{\text{opt}})$ that is the ratio of the preceding volumes.

TABLE 1. Geodesic ball packings above in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ for $\mathbf{p}\mathbf{q}_k\mathbf{o}_\ell$ with $k=1, o=2, \ell=1$.

q	p	ρ_{opt}	$\text{vol}(B(\rho_{\text{opt}}))$	$\text{vol}(\mathcal{P}_p)$	$\delta(\rho_{\text{opt}})$
3	8	0.392699	0.266949	0.411234	0.635408
3	9	0.521044	0.647905	0.822467	**0.787758**
3	10	0.599849	1.017248	1.315947	0.773016
4	5	0.314159	0.134202	0.246740	0.543899
4	6	0.501354	0.573426	0.822467	0.697203
4	7	0.613204	1.092403	1.586186	0.688698
5	4	0.261799	0.076892	0.164493	0.467450
5	5	0.485013	0.516444	0.822467	0.627920
5	6	0.614925	1.102375	1.754596	0.628278

6. GEODESIC BALL PACKINGS IN $\mathbf{S}^2\!\times\!\mathbf{R}$ SPACE

The structure and the model of $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry are described here. We briefly show the discrete isometry groups of the $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry.

The points in the $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry are described by (P,p) where $P\in\mathbf{S}^2$ and $p\in\mathbf{R}$. The isometry group $Isom(\mathbf{S}^2\!\times\!\mathbf{R})$ of $\mathbf{S}^2\!\times\!\mathbf{R}$ can be derived from the direct product of the isometry group of the spherical plane $Isom(\mathbf{S}^2)$ and the isometry group of the real line $Isom(\mathbf{R})$. The structure of an isometry group $\Gamma\subset Isom(\mathbf{S}^2\!\times\!\mathbf{R})$ is the following: $\Gamma=\{(A_1\times\rho_1),\dots(A_n\times\rho_n)\}$, where $A_i\times\rho_i := A_i\times(R_i,r_i) := (g_i,r_i)$, $(i\in\{1,2,\dots n\})$ and $A_i\in Isom(\mathbf{S}^2)$, R_i is either the identity map $\mathbf{1_R}$ of $\mathbf{R}$ or the point reflection $\overline{\mathbf{1}}_{\mathbf{R}}$. $g_i := A_i\times R_i$ is called the linear part of the transformation $(A_i\times\rho_i)$ and r_i is its translation part. The multiplication formula is the following:

$$(A_1\times R_1,r_1)\circ(A_2\times R_2,r_2)=(A_1A_2\times R_1R_2,r_1R_2+r_2). \tag{8.1}$$

A group of isometries $\Gamma\subset Isom(\mathbf{S}^2\!\times\!\mathbf{R})$ is called space group *if the linear parts form a finite group Γ_0 called the point group of Γ. Moreover, the translation components of the identity of this point group are required to form a one-dimensional lattice L_Γ of $\mathbf{R}$.*

In [3] J. Z. Farkas classified and gave the complete list of the space groups in $\mathbf{S}^2\!\times\!\mathbf{R}$.

In [27] we have studied the geodesic balls and their volumes in $\mathbf{S}^2\times\mathbf{R}$, moreover introduced the notion of geodesic ball packing and its density.

In this survey we only recall the top results in the next subsection 6.1 from [24] *where we studied the class of* $\mathbf{S}^2\times\mathbf{R}$ *space groups* **4q. I. 2** *(with a natural parameter* $q \geq 2$*, see* [3]*). Each of them belongs to the glide reflection groups, i.e., the generators* $\mathbf{g}_i$ $(i = 1, 2, \dots m)$ *of its point group* Γ_0 *are reflections and at least one of the possible translation components of the above generators differs from zero (see* [28]*).*

6.1. **A very dense multiply transitive ball packing in $\mathbf{S}^2\times\mathbf{R}$ geometry.** We considered an $\mathbf{S}^2\times\mathbf{R}$ space group (see [3, 27] with point group Γ_0 generated by three reflections $\mathbf{g}_i$ $(i = 1, 2, 3)$

$$(+,\ 0,\ [\,]\ \{(2, 2, q)\}),\ q \geq 2,$$
$$\Gamma_0 = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 - \mathbf{g}_1^2, \mathbf{g}_2^2, \mathbf{g}_3^2, (\mathbf{g}_1\mathbf{g}_3)^2, (\mathbf{g}_2\mathbf{g}_3)^2, (\mathbf{g}_1\mathbf{g}_2)^q).$$

The possible translation parts τ_1, τ_2, τ_3 of the corresponding generators of Γ_0 are derived from the so-called Frobenius congruence relations:

$$(\tau_1, \tau_2, \tau_3) \cong (0, 0, 0),\ \big(0, 0, \frac{1}{2}\big),\ \big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big),\ \big(\frac{1}{2}, \frac{1}{2}, 0\big),\ \big(0, \frac{1}{2}, 0\big),\ \big(0, \frac{1}{2}, \frac{1}{2}\big).$$

If $(\tau_1, \tau_2, \tau_3) \cong\ (0, 0, \frac{1}{2})$ then we have obtained the $\mathbf{S}^2\times\mathbf{R}$ space group **4q. I. 2** (for a fixed q, $2 \leq q \in \mathbf{N}$).

The fundamental domain of the point group of the space group considered is a spherical triangle $A_1A_2A_3$ with angles $\frac{\pi}{q}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ in the base plane. It can be assumed that the fibre coordinate of the centre of the optimal ball is zero and it is a point of the triangle $A_1A_2A_3$.

We consider ball packings related to parameter $q = 2$*.*

In case $K = A_3$

Fig. 6 shows the orbit of the point $K = A_3$ (also K_3) by the space group considered. The images of K lie on a line through the origin and A_3.

$$\phi_3 = \frac{\pi}{4} \approx 0.78539816,\ \ \theta_3 = \frac{\pi}{2} \approx 1.57079633,\ \ R_3 \approx 1.81379936,$$
$$Vol(B(R_3)) \approx 20.00238509,\ \ \delta(R_3, K_3) \approx 0.87757183. \tag{2.12}$$

The "outwardly transformed" images of the balls surround the initial balls (see Fig. 6) thus the touching number of this packing is 4 (see [24]). Finally, we obtain the following

Theorem 6.1 ([24])**.** *The ball arrangement* $\mathcal{B}_{opt}(R_3, K_3)$ *provides the densest multiply transitive ball packing of* $\mathbf{S}^2\times\mathbf{R}$ *space group* **4q. I. 2** $(q = 2)$.

Remark 6.2.

1. To the authors' best knowledge there are no results for the geodesic ball packings in **Sol** geometry at the time of writing.
2. In **Nil**, $\widetilde{\mathbf{SL}_2\mathbf{R}}$ and **Sol** spaces we have studied the so-called *translation ball packings* reported in [18, 17, 19, 30, 34] but we did not consider these cases in this work.

7. The conjecture for the densest ball arrangement in Thurston geometries

We introduced the density function for the geodesic ball packings generated by a discrete group of isometries in a given Thurston geometry. This density is related to the Dirichlet–Voronoi cells generated by the centres of balls. For these ball packings we can formulate the following

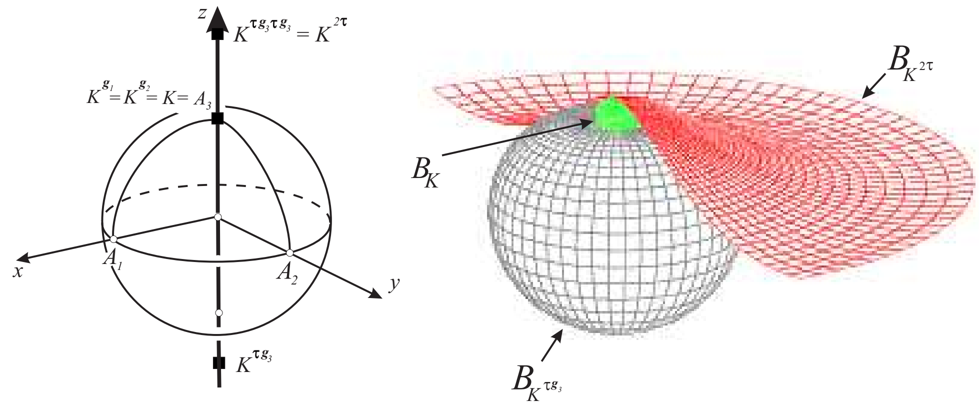

FIGURE 6. a. The orbit of $K = A_3$ by the group $\Gamma = \mathbf{4q.\,I.\,2}$ ($q = 2$ and τ is the *translation part* of the group). b. The densest ball packing is determined by its balls B_K, $B_{K^{\tau g_3}}$ and a part of the sphere $B_{K^{2\tau}}$.

Conjecture 7.1 ([24]). Let $\mathcal{B}$ be an arbitrary congruent geodesic ball packing in a Thurston geometry X (except $\mathbf{S}^3$, where the problem is trivial), where $\mathcal{B}$ is generated by a discrete isometry group of X. The above determined ball arrangement, in $\mathbf{S}^2\!\times\!\mathbf{R}$ $\mathcal{B}_{opt}(R_3, K_3)$ with density $\delta(R_3, K_3) \approx 0.87757183$ provides the densest congruent geodesic ball packing for the Thurston geometries.

The general definition of the density of congruent geodesic ball packings for the Thurston geometries is not settled yet. However, by our investigation for any "good" definition of density the following conjecture may be formulated.

Conjecture 7.2 ([24]). The densest congruent geodesic ball packing in the Thurston geometries is realized by the above ball arrangement $\mathcal{B}_{opt}(R_3, K_3)$ with density $\delta(R_3, K_3) \approx 0.87757183$.

CONCLUSION

In this paper we mentioned only some classical theorems and problems related to Thurston spaces, but we hope that from these the reader can appreciate that our projective method is suitable to study and solve similar problems that represent a huge class of open mathematical problems. Detailed studies are the objective of ongoing research.

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