South Bohemia Mathematical Letters
Volume 24, (2016), No. 1, 10-16.

INTERSECTION OF CURVES – FACTS, COMPUTATIONS, APPLICATIONS IN BLOWUP*

PAVEL CHALMOVIANSKÝ

ABSTRAKT. We deal with application of intersection of curves in resolution of a singularity. The basic tool is a product of curves in homology group used on a certain set of generators producing as a result multiplicity of the infinitely near points of a singularity in the inverse of their proximity matrix. We explain the interconnection of several concepts related to a isolated singularity such as blowup, proximity relations, intersections and homology group of the blowup illustrated on an example.

Importance of intersection

The intersection of varieties is a fundamental tool for their inspection and mutual position. There is a huge theory of intersection covering many cases, see [2]. We restrict to a very special case following [3, 5] and we mention just few important applications in order to illustrate the range of the topic at first. A classical topic is Bézout theorem which was in fact known to earlier authors. Its applications are often starting point of many modern topics in intersection theory such as intersection multiplicity (a generalization of root multiplicity), stability of intersection (Moving lemma). Clearly, the singular points of a curve – $Z(f, \nabla f)$, general case for varieties, is a very important and in general form for arbitrary varieties also very much open field of modern problems and approaches to their solutions, if available. In the area of discrete and constructive geometry, applications in configuration counting (e.g. "How many lines intersect three lines in $P^3(\mathbb{C})$ in general position?") can be easily found and they cannot be easily solved in many cases. The general framework for computing is given by Chow ring and its applications. Multiplicity of a point on a curve is a special case of intersection multiplicity evolved by Fulton, Vogel and Stückrad [4] and many others (see e.g. [1]).

1. Basic notions

Let C be a plane curve given by a polynomial equation $f = 0$. A point $O \in C$ has multiplicity at least m provided all derivatives of order less than m are zero at O. Alternatively, the Taylor polynomial of f at O has the lowest non-zero term at least of degree m.

Additionally, if some derivative of order m is non-zero at O, we say the point O has exactly the multiplicity m. A point with its multiplicity 0 is outside curve, a point with its multiplicity 1 is a regular point of a curve, a point with its multiplicity at least 2 is a singular point of a curve.

Key words and phrases. singularity, intersection, blowup, homology.

* *This paper is republished from the proceedings of the Czech-Slovak Conference on Geometry and Graphics 2016*

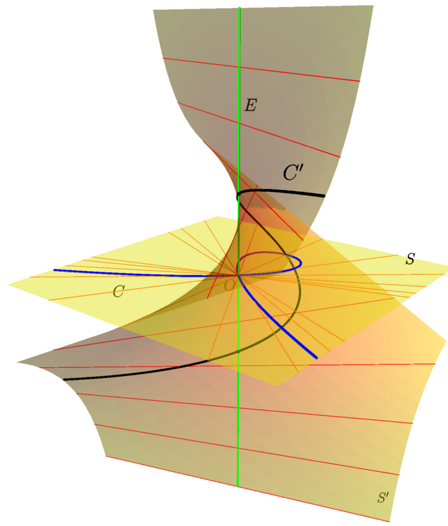


FIGURE 1. The blowup transform of the curve C gives an exceptional divisor E and the strict transform curve C' with two single intersections instead of one double intersection.

As a simple example, we consider a curve given by a polynomial $x^2 \pm y^2$ in an affine plane over complex numbers. It has the point $O = (0,0)$ of multiplicity 2.

Many questions arise with the notion of multiplicity with respect to its computation and its influence on the quality of a curve. The basic cover how many points of certain multiplicity can the curve have, what can be their configurations. These questions are solved form very broad classes of varieties in terms of other invariants.

2. Multiplicity of a point on a curve via blowup

One way of exploring the multiplicity of a point, as a local value, is using blowup transform. We prefer to explain it using fig. 1. A more formal way can be found in many books.

The curve C on the plane S is blown in its singularity O up creating the exceptional divisor E and the strict transform curve C', both on the surface S'. The simple node singularity vanishes, however there is a relation of intersection of E and C' instead respecting the multiplicity of the origin.

Symbolically, if $\pi\colon S' \to S$ is a projection: $\pi^*[C] = 2[E] + [C']$, since the point O is double.

Clearly, it may happen that some points on the curve C', even those on E are again singular. Hence, additional blowup mappings are required.

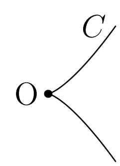
An algebraic curve $C \subset P^2(\mathbb{C}) = S = S_0$ with its resolution $\pi\colon S' \to S$ is given as a composition of a finite sequence of blowups $\pi_i\colon S_{i+1} \to S_i$, $i \in \{0, \dots, N-1\}$ for singular points O_i. The curve $C = C^{(0)}$ is blown up into $C^{(1)}$ and several copies of the divisor E_0, etc. The image of E_i in S_k is denoted $E_i^{(k)}$ (we start always with $E_i^{(i+1)}$).

In some steps, we also look at the considered complex curves/surfaces as the real manifolds. The (complex) curve C is closed oriented (real) 2-dimensional manifold as well as E and C' mentioned above. Here, S', S_i, S are (complex) surfaces, some obtained by the blowup technique.

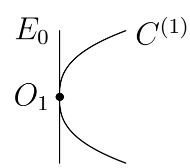
We show certain calculus which is possible within the graph of the blowup. It is based on homology groups of the curve.

Topologically, the blowup can be seen as a gluing of a $\mathbb{S}^2 = P^1(\mathbb{C}) \simeq E_i$ instead of a singular point. Roughly, we get such generator $E_0, \ldots, E_{N-1}$ of the homology group $H_2(S')$ for each singular point during the resolution of the singularity. We demonstrate it on the following example.

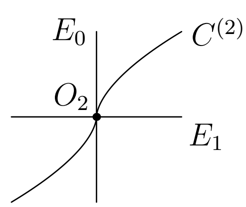
2.1. **Example of resolution.** Starting with the curve $C\colon y^8 = x^{11}$, its singularity O



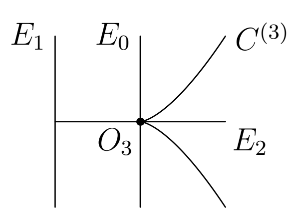
and using the quadratic transform $x_1 y_1 = y$, $x_1 = x$, one gets $(x_1 y_1)^8 = x_1^{11}$, hence $E_0 : x_1^8 = 0$, $C^{(1)}\colon y_1^8 = x_1^3$.



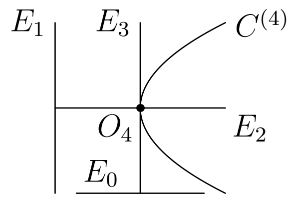
The next blowup $x_2 y_2 = x_1$, $y_2 = y_1$ gives $C^{(2)}\colon y_2^5 = x_2^3$ and $E_1 : y_2^3 = 0$. Clearly, $E_0^{(2)}\colon x_2 = 0$.



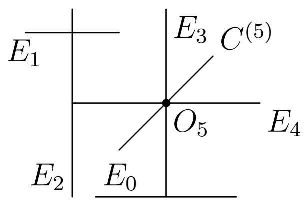
Then, $x_3 y_3 = x_2$, $y_3 = y_2$ provides $C^{(3)}\colon y_3^2 = x_3^3$, $E_2\colon y_3^3 = 0$, $E_0^{(3)}\colon x_2 = 0$, $E_1^{(3)}\colon$ on the other map of a surface due to the fact we work over projective plane/surface.



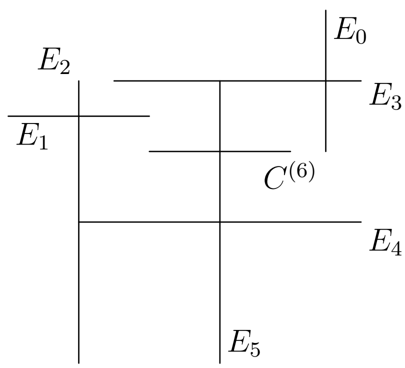
The transform $x_4 = x_3, x_4 y_4 = y_3$ changes the equation into $C^{(4)}\colon y_4^2 = x_4$, $E_3\colon x_4^2 = 0$ and $E_0^{(4)}, E_1^{(4)}\colon$ on the other map of a surface, $E_2^{(4)}\colon y_4 = 0$.



Resolving the tangent exceptional divisor using $x_4 = x_5 y_5$, $y_4 = y_5$ gives $E_4\colon y_5 = 0$, $C^{(5)}\colon y_5 = x_5$, $E_3^{(5)}\colon x_5 = 0$.



And finally, no three divisors intersect and it was resolved with $x_5 = x_6$, $y_5 = x_6 y_6$ resulting in $E_5\colon x_6 = 0$, $C^{(6)}\colon y_6 = 1$, $E_4^{(6)}\colon y_6 = 0$.



2.2. **Graph of the resolution.** The whole resolution process can be described formally by a graph. The vertices of the graph are the singular points O_i of the curve during the whole resolution. The edges are given by proximity relation. We say O_j is proximate to O_i iff O_j is located over O_i at E_i. Clearly, O_i is proximate to O_{i-1} for each $i > 0$.

The whole proximity relation can be written down into a proximity matrix $P = (p_{ij})$ so that

$$p_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } O_j \text{ is proximate to } O_i, \\ 0 & \text{otherwise.} \end{cases}$$

The above example $y^8 = x^{11}$ provides the graph in fig. 2 and the corresponding proximity matrix

$$P = \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

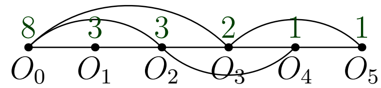


FIGURE 2. The graph of the resolution of a singularity. The vertices are infinitely near points of the singularity and the edges are given by the proximity relation.

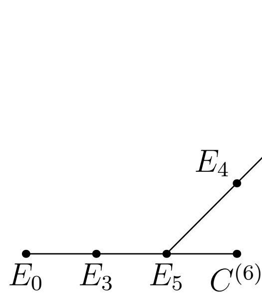


FIGURE 3. The dual graph of the resolved curve $y^8 = x^{11}$.

Clearly, each of the points O_i might be a singular point on the curve on which it occurs. Hence, its multiplicity is interesting as well.

2.3. **Multiplicity of infinitely near points.** The multiplicity $m_i(C)$ is the multiplicity of O_i on the strict transform of C.

One can show directly $m_j(C) = \sum_i m_i(C)$, where O_i is proximate to O_j. The last column of the inverse matrix to the proximate matrix P contains these numbers

$$P^{-1} = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & 8 \\ 0 & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Dual graph of the resolution

Now, we get another description of the singularity using dual graph. The vertices of the dual graph $\Gamma(C)$ are the exceptional divisors E_i and two such vertices share an edge iff they intersect. An additional natural subdivision rule for edges is used in case O_i is proximate to O_{i-1} and some O_j for $j < i - 1$.

If the curve has several branches, it is convenient to add vertices for each resolved branch of the curve to each corresponding E_N which is crossed transversely. The resulting graph is called augmented dual graph of the curve. For the above example, the dual graph is in fig. 3.

The shape of the dual graph cannot be arbitrary. A typical dual graph looks like the one in fig. 4. This is due to Euclid's algorithm which is encoded in the resolution (on the exponents of the curve at singularity) of the branch and certain subdivision rule as we could have seen in the above example.

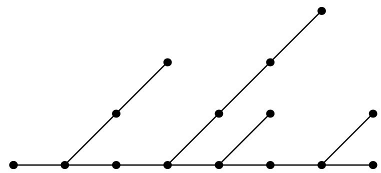


FIGURE 4. Typical dual graph of an isolated singularity

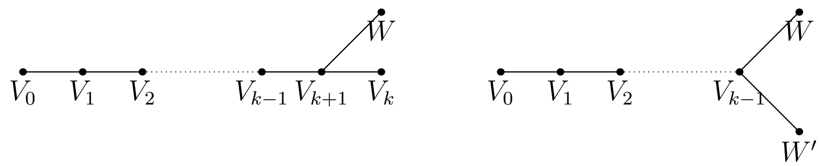


FIGURE 5. The dual graphs of A_n singularities. The graph for A_{2k} and A_{2k+1}.

Structurally, the simplest singularities are those name A_n, $n \geq 1$. There are two cases, $n = 2k$, $n = 2k - 1$. The corresponding dual graphs are drawn in the fig. 5.

4. Computation in homology groups

The computations of the multiplicity can be interpreted in terms of homology group of the blown up curve via intersection.

Let $\pi\colon S' \to S$ be a blowup of $C \subset S$ given by $f = 0$. Let C' be a strict transform of C, O has multiplicity m on C and E be the corresponding exceptional divisor. Then, in the group $H_2(T)$

$$\pi^*[C] = [C'] + m[E],$$

moreover, one has the additional relations

$$\begin{aligned}[E].[E] &= 1,\\ [E].\pi^*[F] &= 0,\\ \pi^*[F].\pi^*[G] &= [F].[G]\end{aligned}$$

for any g such that $[G] = [Z(g)]$ and $[F] = [Z(f)]$.

What is a geometric meaning of $[E].[E] = -1$? The intersection form counts the intersection points of the corresponding curves with orientation. The above fact says, we cannot move freely the exceptional curve on the blowup surface so that the original and the moved one intersect with the same orientation as e.g. lines in the projective complex plane do.

Let $[\hat{E}_j]$ be the strict transform of $[E_j]$ in the resolution. It can be shown by induction that

$$[E_i] = \sum_j p_{ij}[\hat{E}_j]$$

since $\pi_j([E_i^{(j)}]) = [E_i^{(j+1)}] - p_{ij}[E_j^{(j+1)}]$.

This provides another look at the multiplicity of a point. It is the intersection number of the strict transform of the curve with the strict transform of the exceptional divisor corresponding to the point. Hence, the inversion of the proximity matrix gives the sought intersection numbers.

Conclusion

We reviewed several concepts of a multiplicity of a point using blowup, graphs, homology and other modern tools of algebraic geometry. As an example, we used a curve which has a cusp of higher order as a singularity to demonstrate the concepts.

References

[1] Eisenbud, D., Harris, J.:*3264 and All That Intersections: A Second Course To Algebraic Geometry*. Cambridge University Press, Cambridge, 2016.

[2] Fulton, W.:*Intersection theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge; A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* (Vol. 2). Springer-Verlag, Berlin, second edition, 1998.

[3] Greuel, G.-M., Lossen, C. and Shustin, E.:*Introduction to singularities and deformations*. Springer, Berlin, 2007.

[4] Vogel, W.:*Lectures on results on Bezout's theorem; Tata Institute of Fundamental Research Lectures on Mathematics and Physics* (Vol. 74). Springer-Verlag, Berlin, 1984.

[5] Wall, C. T. C.:*Singular points of plane curves; London Mathematical Society Student Texts* (Vol. 63).Cambridge University Press, Cambridge, 2004.

Department of algebra, geometry and math education, Comenius University in Bratislava. Bratislava, Slovakia

E-mail address: pavel.chalmoviansky@fmph.uniba.sk