South Bohemia Mathematical Letters
Volume 25, (2017), No. 1, 39-47.

REAL-WORLD APPLICATIONS OF NUMBER THEORY

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Dedicated to the eminent Czechoslovak mathematician Ladislav Skula

Abstract. The present paper is concerned with practical applications of the number theory and is intended for all readers interested in applied mathematics. Using examples we show how human creativity can change the results of the pure mathematics into a practical usable form. Some historical notes are also included.

1. Introduction

German mathematician Johann Carl Friedrich Gauss (30 April 1777 - 23 February 1855), regarded as one of the greatest mathematicians of all time, claimed: "*Mathematics is the queen of the sciences and number theory is the queen of mathematics.*" However, for many years number theory had only few practical applications. It is well known that the great English number theorist Godfrey Harold Hardy (7 February 1877 - 1 December 1947) believed that number theory had no practical applications. See his essay *"A Mathematician's Apology"* [16]. Over the 20th and 21st centuries, this situation has changed significantly. Contrary to Hardy's opinion, many practical and interesting applications of number theory have been discovered. The present paper brings some remarkable examples of number theory applications in the real world. The paper can be regarded as a loose continuation of the author's preceding work [19] and [20].

2. Diophantine equations

Diophantine analysis is a branch of the theory of numbers studying polynomial equations in two or more unknowns which are to be solved in integers. The equations themselves are called Diophantine. Note, that the name Diophantine refers to the Greek mathematician Diophantus of Alexandria who lived in the third century B.C. Finding solutions of polynomial equations in integers is one of the oldest mathematical problems. Traditionally, the following basic questions are solved:

(i) Find whether a given Diophantine equation has at least one integer solution.
(ii) Decide whether the number of integer solutions is finite or infinite.
(iii) Establish all integer solutions of a given Diophantine equation.

It is also natural to ask whether there is an algorithm that will find the solutions to any given Diophantine equation. This question is known as Hilbert's tenth problem. In 1970, Russian mathematician Yuri Vladimirovich Matiyasevich [24]

2010 *Mathematics Subject Classification.* 00A69, 11D45, 05A17, 01Axx.
Key words and phrases. number theory, applications, Diophantine equations, partitions, history.

showed that such a general algorithm does not exist. However, for many specific Diophantine equations, the general algorithm is well known. As an example, the theory of linear Diophantine equations can be given.

Let n be a positive integer, $n \geq 2$. Then, the equation

$$a_1x_1 + \cdots + a_nx_n = m \tag{2.1}$$

is said to be a linear Diophantine equation if all unknowns $x_1, \ldots, x_n$ and all coefficients $a_1, \ldots, a_n, m$ are integers. It is well known that an integer solution of (2.1) exists if and only if the greatest common divisor of $a_1, \ldots, a_n$ divides m. For general methods for solving (2.1), see for example [5], [25], and [27, pp. 27–31].

In the following sections we give three interesting examples of using Diophantine equations in the natural sciences.

3. Balancing of chemical equations

As the first example we show some application of a linear Diophantine equation to problems in chemistry. In particular, we will deal with the balancing of chemical equations. See [6]. Consider a chemical equation written in the form

$$x_1A_{a_1}B_{b_1}C_{c_1}\cdots + x_2A_{a_2}B_{b_2}C_{c_2}\cdots + \cdots \to x'_1A_{a'_1}B_{b'_1}C_{c'_1}\cdots + x'_2A_{a'_2}B_{b'_2}C_{c'_2}\cdots + \cdots$$

where $A, B, C, \cdots$ are the elements occurring in the reaction, $a_1, b_1, c_1, \cdots, a'_1, b'_1, c'_1, \cdots$ are positive integers or 0, and $x_1, x_2, \cdots, x'_1, x'_2, \cdots$ are the unknown coefficients of the reactants and products. Then, we have

$$\begin{aligned} x_1a_1 + x_2a_2 + \cdots &= x'_1a'_1 + x_2a'_2 + \cdots \\ x_1b_1 + x_2b_2 + \cdots &= x'_1b'_1 + x_2b'_2 + \cdots \\ x_1c_1 + x_2c_2 + \cdots &= x'_1c'_1 + x_2c'_2 + \cdots \\ &\cdots \end{aligned} \tag{3.1}$$

Clearly, each equation of (3.1) expresses the law of conservation of the number of atoms for any particular element $A, B, C, \cdots$. Finding all integer solutions $[x_1, x_2, \cdots, x'_1, x'_2, \cdots]$ of (3.1) is a nice elementary problem of Diophantine analysis.

We show a concrete example. Let us consider the chemical equation

$$x_1\mathrm{KMnO_4} + x_2\mathrm{H_2SO_4} + x_3\mathrm{H_2O_2} \to x'_1\mathrm{K_2SO_4} + x'_2\mathrm{MnSO_4} + x'_3\mathrm{H_2O} + x'_4\mathrm{O_2}. \tag{3.2}$$

¿From (3.2) we immediately obtain

$$\begin{aligned} 4x_1 + 4x_2 + 2x_3 &= 4x'_1 + 4x'_2 + x'_3 + 2x'_4 && \text{for O} \\ x_1 &= x'_2 && \text{for Mn} \\ x_1 &= 2x'_1 && \text{for K} \\ x_2 &= x'_1 + x'_2 && \text{for S} \\ 2x_2 + 2x_3 &= 2x'_3 && \text{for H} \end{aligned} \tag{3.3}$$

This system is easily reduced to

$$5x_1 + 2x_3 - 4x'_4 = 0. \tag{3.4}$$

Clearly, (3.4) is a linear Diophantine equation in three variables with a solution $[x_1, x_3, x'_4] = [2, 1, 3]$. Hence, $[x_1, x_2, x_3, x'_1, x'_2, x'_3, x'_4] = [2, 3, 1, 1, 2, 4, 3]$. Consequently,

$$2\mathrm{KMnO_4} + 3\mathrm{H_2SO_4} + \mathrm{H_2O_2} \to \mathrm{K_2SO_4} + 2\mathrm{MnSO_4} + 4\mathrm{H_2O} + 3\mathrm{O_2}. \tag{3.5}$$

It is evident that (3.5) is not the only solution of our balancing problem. In fact, after a short calculation, we see that the set S of all positive integer solutions of (3.3) is infinite and can be written in the form

$$S = \{[2u, 3u, v, u, 2u, 3u + v, (5u + v)/2] : u, v, (5u + v)/2 \in \mathbb{N}\}. \tag{3.6}$$

Observe now that the solution (3.5) can be obtained from (3.6) by putting $u = v = 1$. Hence, (3.5) is the smallest possible solution of the balancing problem (3.2). Finally, we see that $(5u + v)/2 \in \mathbb{N}$ if and only if $u \equiv v \pmod 2$. Hence, it readily follows that S can be written in the form $S = S_1 \cup S_2$ where

$$S_1 = \{[4r - 2, 6r - 3, 2s - 1, 2r - 1, 4r - 2, 6r + 2s - 4, 5r + s - 3] : r, s \in \mathbb{N}\}$$

and,

$$S_2 = \{[4r, 6r, 2s, 2r, 4r, 6r + 2s, 5r + 2] : r, s \in \mathbb{N}\}.$$

For further examples of balancing equations see R. Crocker [6, p. 732].

4. Determination of the molecular formula

In this section we show how linear Diophantine equations can be used to determine the molecular formula [6]. Assume that a substance with a molecular weight of m contains elements $A, B, C, \cdots$ with atomic weights $a, b, c, \cdots$ and that $x, y, z, \cdots$ represent the numbers of atoms of $A, B, C, \cdots$ in a molecule. Then, we have

$$ax + by + cz + \cdots = m. \tag{4.1}$$

Let $\alpha, \beta, \gamma, \cdots$ denote the integers nearest the values $a, b, c, \cdots$ and μ denote the integer nearest m. Then, (4.1) can be replaced by the linear Diophantine equation

$$\alpha x + \beta y + \gamma z + \cdots = \mu. \tag{4.2}$$

If we require that the values $x, y, z, \cdots$ in (4.2) should be reasonably small, we can solve (4.2) under a condition

$$-\frac{1}{2} < (a - \alpha)x + (b - \beta)y + (c - \gamma)z + \cdots < \frac{1}{2}. \tag{4.3}$$

If more solutions of (4.2) are obtained, the true values may be found by substituting into (4.1) and finding which of them satisfies (4.1) with minimum deviation from m.

The following problem will be now solved: *The molecular weight of a substance containing only hydrogen and sulfur is* 66.146. *What is the molecular formula?*

Let a denote the atomic weight of hydrogen and b the atomic weight of sulfur. Using the periodic table of elements, we find that $a = 1.008$ and $b = 32.065$. Hence, we have $1.008x + 32.065y = 66.146$. Next, we see that $\alpha = 1$, $\beta = 32$, $\mu = 66$ and that $x \le 34$, $y \le 2$. Subject to these conditions, it is easy to obtain that the Diophantine equation $x + 32y = 66$ has only two positive integer solutions $[x, y] = [34, 1]$ and $[x, y] = [2, 2]$. Since a molecule of this size is not likely to contain 34 hydrogen atoms and 1 sulfur atom, this possibility may be eliminated. Therefore, $[x, y] = [2, 2]$ and, the resulting molecular formula is H_2S_2. However, in solving this problem, we can proceed in a more efficient way. The equation $1.008x + 32.065y = 66.146$ can be converted to the Diophantine equation $1008x + 32065y = 66146$, which has infinitely many integer solutions $[x, y] = [2 + 32065 \cdot k, 2 - 1008 \cdot k]$, $k \in \mathbb{Z}$. Since $x, y \in \mathbb{N}$ and $x \le 34$, $y \le 2$, the solution $[x, y] = [2, 2]$ immediately follows.

5. Structure of viruses

In this section we focus on an interesting problem in virology. Recall, that virus particles consist of protein subunits ordered geometrically according to strict symmetry rules. These rules highly depend on the chemical properties of the protein. For example, it is well known that spherical viruses prefer the icosahedral symmetry and that the total number N of nearly identical subunits that may be regularly ordered on the closed icosahedral surface is given by Goldberg's formula [8]

$$N = 10(a^2 + ab + b^2) + 2 = 10T + 2, \text{ where } a, b \in \mathbb{N} \cup \{0\}. \tag{5.1}$$

Using (2.11) we readily find, that

$$N \in \{12, 32, 42, 72, 92, 122, 132, \cdots\}.$$

On the other hand, it is known that an icosahedron has 30 axes of twofold symmetry, 20 axes of threefold symmetry and 12 axes of fivefold symmetry. Therefore, all subunits on the surface of an icosahedral virus may be divided into 30 identical groups each having a twofold symmetry, 20 groups with threefold symmetries and 12 groups with fivefold symmetries. These groups are often called disymmetrons, trisymmetrons and pentasymmetrons, respectively. Assume now that any disymmetron contains d_u subunits, any trisymmetron contains t_v subunits and any pentasymmetron contains p_w subunits. Then, by [22], we have

$$N = 30d_u + 20t_v + 12p_w = 10T + 2, \tag{5.2}$$

where

$$d_u = u - 1,\ t_v = \frac{(v-1)v}{2},\ p_w = \frac{5(w-1)w}{2} + 1 \quad \text{and,} \quad u, v, w \in \mathbb{N}. \tag{5.3}$$

For each value of N defined by (5.1), the number $f(N)$ of all the solutions of (5.2) corresponds to the number of theoretically possible ways of making a virus with N subunits, but with different combinations of symmetrons. For example, if $N = 42$, then (5.2) has the unique solution $42 = 30 \cdot 1 + 20 \cdot 0 + 12 \cdot 1$, if $N = 72$, then (5.2) has exactly three solutions: $72 = 30 \cdot 2 + 20 \cdot 0 + 12 \cdot 1 = 30 \cdot 0 + 20 \cdot 3 + 12 \cdot 1 = 30 \cdot 0 + 20 \cdot 0 + 12 \cdot 6$.

Putting $x = 2v - 1$, $y = 2w - 1$, $z = u - 1$ and using (5.3) equation (5.2) can be transformed, after some calculations, to the equivalent form

$$x^2 + 3y^2 + 12z = 4T. \tag{5.4}$$

In this way, the problem of describing the structure of viruses by means of geometric symmetries is reduced to the following Diophantine problem:

Find all odd positive integers x, y and all non-negative integers z, satisfying $x^2 + 3y^2 + 12z = 4(a^2 + ab + b^2)$ for any given values $a, b \in \mathbb{N} \cup \{0\}$.

There is no simple solution to this problem. In [22], W. Ljunggren proved that the total number $f(N)$ of solutions of (5.4) is equal to

$$f(N) = \frac{\pi\sqrt{3}}{180} N + k\sqrt{N}, \tag{5.5}$$

where the number k is bounded and independent of N. Furthermore, from (5.5) it can be easily deduced that $f(N)$ increases linearly with N. Surprising is that this increase is bi-modal. Geometrically, this means that, if $[x, y, z]$ is any solution of

(5.4), then $[x, y]$ lies in the neighbourhood of exactly one of two lines $y = 0.03x$ and $y = 0.015x$. A detailed analysis of this fact can be found in [22, pp. 54–56].

In [10] A. Grytczuk presented an effective method for determining all solutions of (5.4) in odd positive integers x, y and non-negative integers z. Moreover, in [11] A. Grytczuk and K. Grytczuk proved that (5.4) can be reduced to the form

$$x^2 + 3y^2 = 4(R^2 + 3S^2), \quad (R, S) = 1 \tag{5.6}$$

and that all solutions of (5.6) in odd positive integers x, y are given by the formulas

$$x = |R - 3S|, y = R + S \quad \text{or} \quad x = |R + 3S|, y = |R - S|. \tag{5.7}$$

Consequently, (5.7) gives the full solution of our Diophantine problem.

Note that, in fact, the solution (5.7) has been established earlier by G. Xeroudakes. Consult [30, p. 102]. Finally, a mathematical description of some viruses, using the above theory, can be found in [12] and [13]. In particular, parvovirus $T = 1, N = 12$, poliovirus $T = 3, N = 32$, togavirus $T = 4, N = 42$, reovirus $T = 13, N = 132$, herpesvirus $T = 16, N = 162$, and adenovirus $T = 25, N = 252$ are studied in detail and their geometrical models are presented.

6. Partitio numerorum and quantum physics

A partition of a natural number n is any non-increasing sequence of natural numbers whose sum is n. The number of partitions of n is denoted by $p(n)$. For example, if $n = 5$ then, $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Hence, $p(5) = 7$. The problem of establishing the number $p(n)$ has a very long history and it is known under the name of *partitio numerorum*. Since 1674, when the problem was first mentioned by Gottfried Wilhelm Leibniz (1 July 1646 - 14 November 1716), many results concerning $p(n)$ have been discovered. For the basic theory of $p(n)$, see the books [2] and [17, pp. 361–392]. Some recent results on $p(n)$ can be found in the author's paper [18].

For small values of n, it can be found readily that

$$\{p(n)\}_{n=1}^{\infty} = \{1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, \cdots\}.$$

About 1916, Percy Alexander MacMahon (26 September 1854 - 25 December 1929) established the values of $p(n)$ for all n up to 200 [15, pp. 114-115]. For example, he found that

$$p(100) = 1905692292 \quad \text{and} \quad p(200) = 3972999029388.$$

In 1934, H. Gupta [14] extended MacMahon's table up to $n = 300$ and later, in 1937, up to 600. For further historical notes, see [17, p. 391]. Nowadays, using a computer, we can establish that

$$p(1000) = 24061467864032622473692149727991 \approx 2.40615 \cdot 10^{31}$$

and

$$p(10000) \approx 3.61673 \cdot 10^{106}.$$

As we see, the growth of $p(n)$ is very rapid. It is, therefore, natural to ask about the size of $p(n)$. The answer to this question is given by the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{for} \quad n \to \infty, \tag{6.1}$$

which shows that the growth of $p(n)$ is subexponential. The formula (6.1) was discovered in 1917 by G. H. Hardy and the brilliant Indian mathematician Srinivasa Ramanujan (22 December 1887 - 26 April 1920). For a proof of (6.1) see [15]. It is remarkable that the formula (6.1) is extremely accurate and has found important applications in physics. Two interesting connections between the problem *partitio numerorum* and physics will now be mentioned.

First recall that the Hardy-Ramanujan formula has been used, with great success, in quantum physics. The connection between the theory of partitions and quantum physics was first discovered by Niels Henrik David Bohr (7 October 1885 - 18 November 1962) and talented physicist Fritz Kalckar (13 February 1910 - 6 January 1938) in their famous paper [4]. In [4], using Ramanujan - Hardy formula (6.1), Bohr and Kalckar achieved a crucial breakthrough in quantum physics: they described the decomposition of heavy atomic nuclei. Later Bohr pointed out the connection between the decomposition of Uranium 235 with the theory of partitions of natural numbers and the main idea of the nuclear bomb was clearly indicated. In this sense, the ideological creator of the nuclear bomb was Niels Bohr [23, p. 249].

The second very important application of Hardy-Ramanujan formula can be found in the problems of statistical mechanics. The significant role of (6.1) in this branch has been discussed by many authors. See, for example, the papers of C. Van Lier and G. E. Uhlenbeck [29], F. C. Auluck and D. S. Kothari [1], N. H. V. Temperly [28] and, L. Debnath [7]. Now we will give some details to one of these problems. In quantum theory, a boson is a particle that satisfies Bose-Einstein statistics. Examples of bosons are particles such as photons, gluons, W and Z bosons and the recently discovered Higgs boson. For basic definitions see [9, pp. 74-78].

Let us now consider a quantum system of N identical bosons. It is well known that such system can be viewed as a collection of one-dimensional harmonic oscillators. The energy levels of a quantum harmonic oscillator are determined by the equation $E_k = (k + 1/2)\hbar\omega$ where k is non-negative integer, $h = 2\pi\hbar$ is the Planck constant and ω is the angular frequency. For $k = 0$, we obtain the so-called ground state energy and, for $k = 1, 2, \cdots$, we get the excited states. Hence, in the ground state of the system, all bosons occupy the lowest level with $k = 0$. When an excitation energy is given to the system, there are many ways in which this energy can be distributed among N bosons. The fundamental problem is now to determine this number. In fact, this problem is the same as that of finding the number $p(n)$. This follows from the fact that the indistinguishability of boson particles is equivalent to the property that the order of summands is not significant in partitions.

Let us denote by $w(N, n)$ the number of all possible ways of distributing among N bosons the exciting energy $E = n\hbar\omega$. If $N \geq n$, then $w(N, n) = p(n)$ and, for $1 < N < n$, we have $w(N, n) = p_N(n)$ where $p_N(n)$ is the number of partitions of n into exactly N or less than N parts. Consequently, the asymptotic form of $w(N, n)$ for $N \geq n$ is precisely the Hardy-Ramanujan formula (6.1).

Now we explain, using a short example, the basic idea of the correspondence between the number $p(n)$ and the number $w(N, n)$ of states of quantum system of N bosonic harmonic oscillators. Assume that $N = 6$ and $n = 4$. Then, we have $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, which yields $p(4) = 5$. Consider now all possible realizations of the state with energy $E = 4\hbar\omega$ in the system of six harmonic oscillators. Clearly, there are exactly five ways (W1 – W5) to achieve the energy $E = 4\hbar\omega$: (W1) to put one boson into excited state with $k = 4$, (W2) to put one boson to the state with $k = 3$ and one boson into the state $k = 1$, (W3) to put two bosons to the state with $k = 2$, (W4) to put one boson to the state with $k = 2$ and two bosons to the state with $k = 1$, (W5) to put four bosons into the state $k = 1$. All remaining non-excited bosons in (W1-W5) remain in the ground state $k = 0$.

In the below figure, the correspondence between $p(4)$ and $w(6, 4)$ considered will be represented graphically.

(6.2)

	4	$3+1$	$2+2$	$2+1+1$	$1+1+1+1$
$k = 4$	•				
$k = 3$		•			
$k = 2$			••	•	
$k = 1$		•		••	••••
$k = 0$	•••••	••••	••••	•••	••
	W_1	W_2	W_3	W_4	W_5

Figure 1.

Readers interested in the relationship between statistical mechanics and the problem of *partitio numerorum* will find large lists of references in [7], [23], and [26].

7. Concluding remarks

Finally, some further significant applications of the number theory will be shortly mentioned. Above all, it is well known that the theory of Fibonacci numbers has many applications in physics, chemistry, biology, economy, and architecture. Listing 163 chronological references to papers published from 1611 to 2011, paper [19] can serve as an introduction to this field. Further fields of number theory with important applications include the theory of sequences over finite fields [20]. This theory found an application in the testing of Einstein's general relativity or in testing the global warming of oceans. Furthermore, using methods of elementary number theory, practical problems have been solved concerning to the splicing of telephone cables [21]. Many further interesting applications can be found in the book *Number Theory and the Periodicity of Matter* [3]. Lastly, new attractive applications of the number theory include cryptography, coding theory, and random number generation. With the rise of computers, these fields develop very rapidly with their importance continuously increasing.

Acknowledgment. The author would like to thank the anonymous referee for carefully reading the manuscript, and for the helpful comments and suggestions.

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