

## GEOGEBRA3D

ANDREAS LINDNER

**ABSTRACT.** Originally GeoGebra was designed as a program for the dynamic combination of geometry and algebra. Over time additional modules were added such as for example spreadsheet and a computer algebra system (CAS). Currently work is proceeding on an extension for a 3D module which allows the representation of objects in a three-dimensional coordinate system. The currently available beta version (November 2013) is already well advanced and provides an insight into the future version 5.0.

The paper gives a brief introduction to the operation of the program and then displays a few worksheet that have been created on the subjects of geometry, analysis, combination of geometry and analysis, and applied mathematics in science.

### 1. INTRODUCTION - HANDLING OF THE SOFTWARE

The interface of the program is displayed when you first open GeoGebra 5.0 Beta in the usual form with algebra and graphics view. The changes in the new version are only apparent when you click the new command *Graphics View 3D* in the *View menu*. Thus, an additional window will open with a three-dimensional coordinate system that allows to work with GeoGebra in the usual way for three-dimensional depictions.

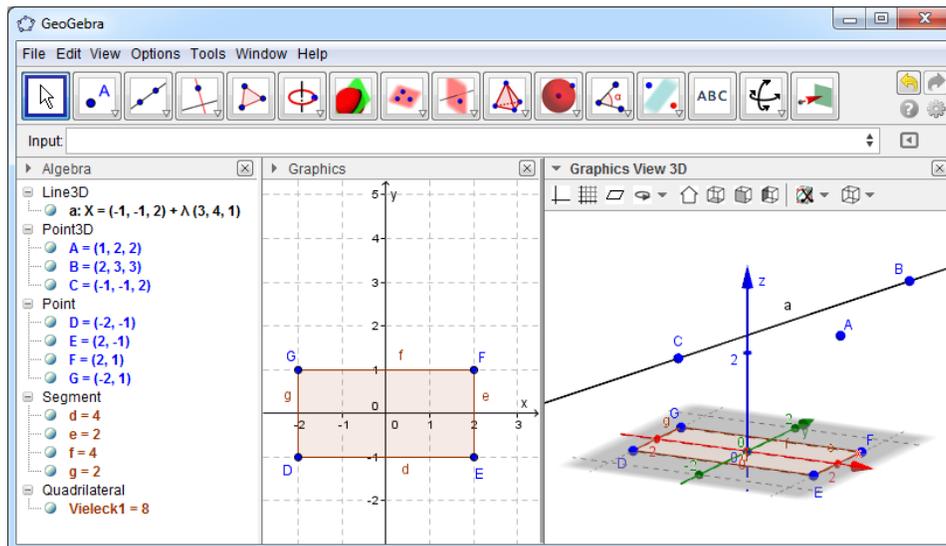


FIGURE 1. GeoGebra 3D with Algebra View, Graphics View and Graphics View 3D

*Key words and phrases.* GeoGebra, GeoGebra3D, GeoGebra 5.0, spatial geometry, three-dimensional geometry.

When changing the windows, the appearance of the toolbar will change as well because a number of new symbols have been added to the Graphics View 3D. The list of tools shows the current state of development of the beta version and may change by the time version 5.0 is completed.

In the 3D module, points can be typed in as usual by using the command line – for example  $A = (1, 2, 3)$  – or can be set by mouse click. After a single mouse click, the points can be moved in the  $xy$ -plane, and a second click of the mouse allows the user to move the point in  $z$ -direction.

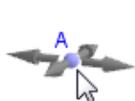


FIGURE 2. Horizontal shifting



FIGURE 3. Vertical shifting

The graphics commands provided in GeoGebra so far - such as line, segment, polygon, etc. - can be used in the Graphics View3D in an analogous manner. This intuitive handling of the program operation is intended to simplify entry to the 3D version and should enable users to switch to the new module very easily.



FIGURE 4. Toolbar of the Graphics View 3D

In GeoGebra, the individual modules are dynamically linked. If a change is made in one module (e.g. in the graphics window), this also has an impact on all of the other modules. Likewise, GeoGebra allows the dynamic interplay of the two-dimensional graphics window with the three-dimensional. Figure 1 shows a rectangle shown in the graphics view and also in the Graphics View 3D.

The design bar in the Graphics View 3D is an important control element and allows rapid switching between different views.



FIGURE 5. Design bar of the Graphics View 3D

The elements of the design bar in detail:

-  Rotation: performs a rotation about the  $z$ -axis with variable speed
-  View towards  $xy$ -plane: rotates the construction into the horizontal projection
-  View towards  $xz$ -plane: rotates the construction into the vertical projection
-  View towards  $yz$ -plane: rotates the construction into the side projection
-  Parallel projection

-  Perspective projection
-  Projection for glasses (Anaglyph 3D)
-  Oblique projection

By changing the views an object can be displayed in different ways.

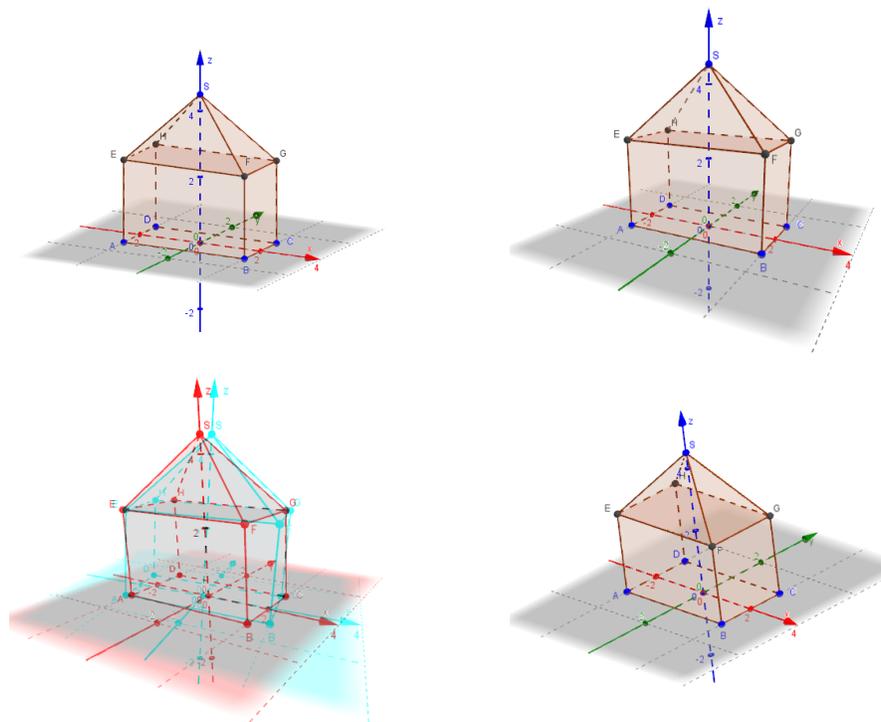


FIGURE 6. The different projections in GeoGebra 3D: Parallel projection (top left), Perspective projection (top right), Anaglyph 3D (bottom left), Oblique projection (bottom right)

The algebraic representation of three-dimensional geometric objects is a logical continuation of the two-dimensional way. Lines are displayed in parameter form, for example as  $g : X = (1, 2, 3) + \lambda(2, -1, 1)$ , planes in the form  $\varepsilon : 3x + y - 2z = 1$  and spheres as  $k : (x - 1)^2 + (y - 2)^2 + (z + 1)^2 = 9$ . This also allows a computational treatment of a problem in the CAS of GeoGebra. Especially in terms of analytical geometry this represents a powerful tool and allows a combination of computational treatment and geometrical solution of a task.

After this brief introduction to the program, some examples of the use of GeoGebra3D will be presented subsequently.

## 2. GEOMETRY

**Geometry 1: True Length of a Line Segment [1]**

The construction of the true length of a segment is one of the basic tasks in a course on spatial geometry. The applet will illustrate why a segment appears shortened in a projection and clarify the construction method for the true length. By rotating the coordinate system to an appropriate location, the segment can also be viewed in its true length.

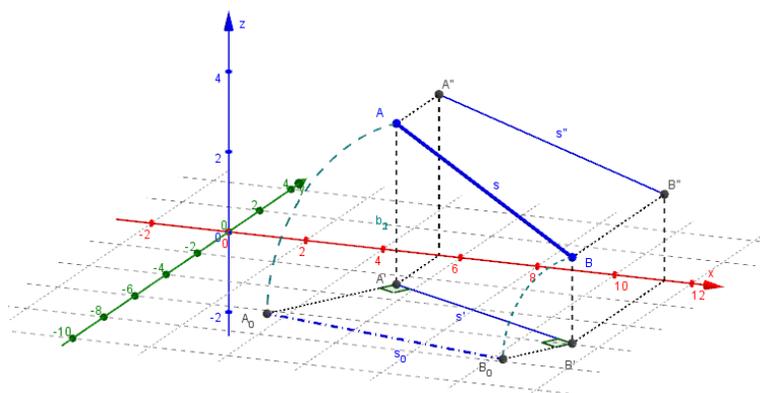


FIGURE 7. Segment in true length

This worksheet is designed to help the user understand the creation of horizontal and vertical projection. Points A and B of the segment are moveable.

**Geometry 2: Minimal Distance of two Skew Lines [2]**

The minimal distance between the two skew straight lines is that segment which has the minimum distance between the two straight lines. It is perpendicular to both lines.

In addition to the straight lines and the minimal distance the construction shows two more planes, each formed by a line and the minimal distance. So an enhanced spatial effect is obtained and the design appears three-dimensional. The rotation of the coordinate system allows the user to view the straight lines from various angles. In this case the tool *View in front of* , which gives a view perpendicular to a selected object, is very helpful. If this tool is applied to one of the straight lines, the construction is rotated such a way that the straight line is projected as a point and the minimal distance appears in true length.

The points  $A, B, C$  and  $D$  are dynamically variable so that the position of the minimal distance can be simulated for various options. Using the CAS of GeoGebra, the points  $G$  and  $H$  and the length of the minimal distance can also be calculated by means of analytical geometry.

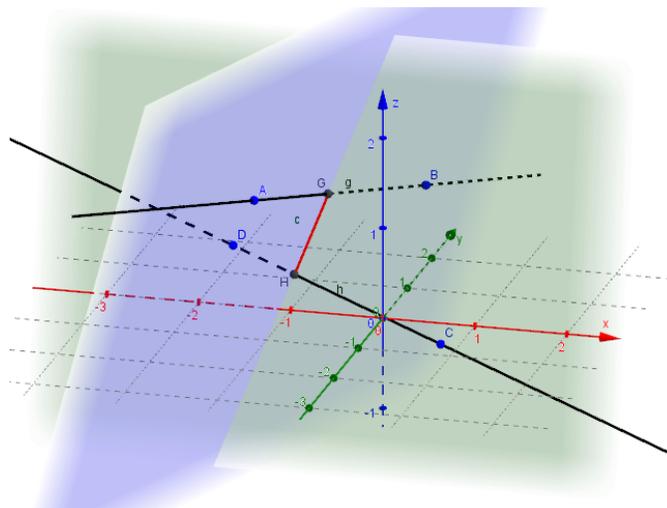


FIGURE 8. Minimal distance of two skew lines

### Geometry 3: Analytic Geometry: Pyramid [3]

The next example shows how GeoGebra3D can be profitably used for analytical geometry.

*Task: The points  $A(6|1|12)$ ,  $B(-6|-2|9)$ ,  $C(-2|-7|-2)$  and  $D$  are the base of a right pyramid with a square base. a) Determine  $D$  as an intersection point of the plane  $\varepsilon : 2x + y - 4z = 12$  and the straight line  $g : X = (6|-2|1) + t \cdot (2|-1|0)$ . Prove that  $D$  forms a square with  $A$ ,  $B$  and  $C$ . b) Determine the coordinates of the top  $S$  of the pyramid if the height is 9 LE (2 ways).*

By means of a simultaneous use of CAS and Graphics View 3D the problem is solved geometrically and computationally, with the results of the geometric solution shown in the algebra view.

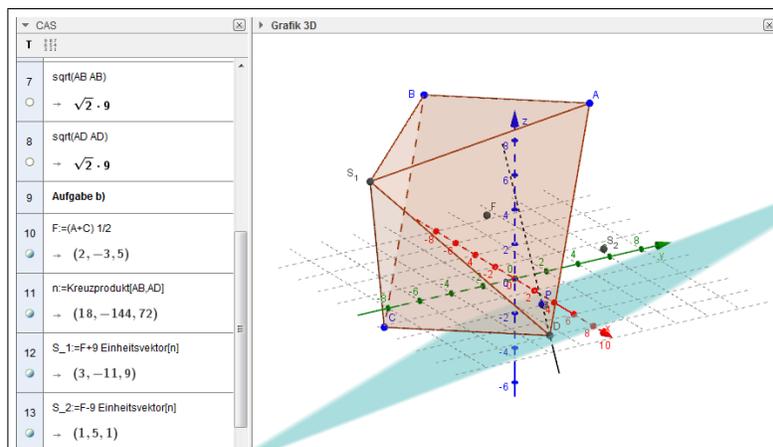


FIGURE 9. Calculation and construction of a pyramid

### Geometry 4: Analytic Geometry: Hexagonal Prism and Pyramid

The task in the next example is: *A sphere is inscribed in a six-sided prism. Where should the vertex of a pyramid be when the pyramid is placed on the prism and touches the sphere?*

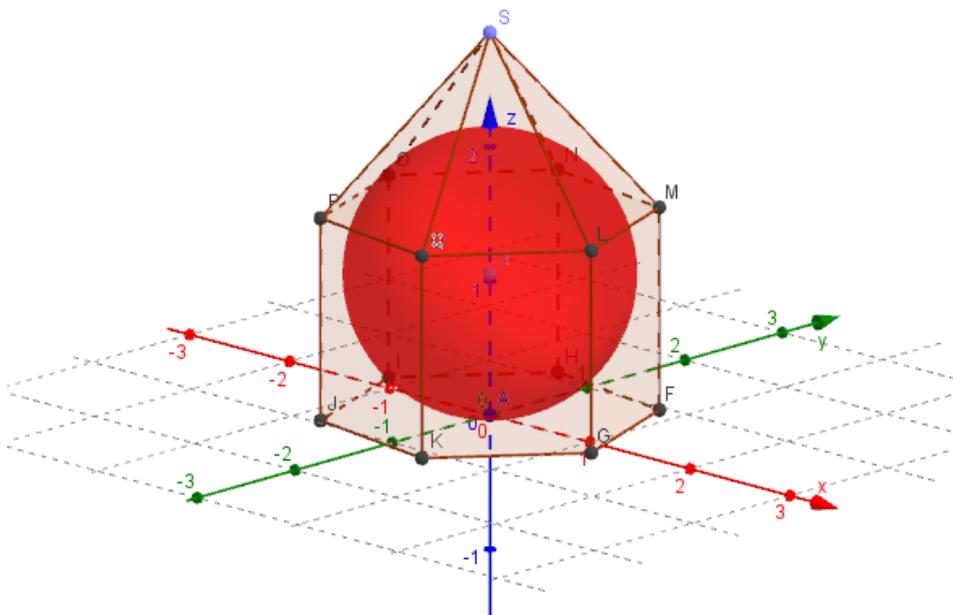


FIGURE 10. Six-sided prism with pyramid and sphere

In this case the method of construction is shown in a video that was recorded with a screen recorder while working with GeoGebra, which makes the individual design steps comprehensible.

Lately it can be observed that more and more often software manufacturers, textbook authors and also teaching tutorials use videos. This format is based on the everyday needs of students who often appreciate videos much more than illustrations in manuals in print format. The video was provided to the author by the programmer of GeoGebra3D module, Mathieu Blossier.

### Geometry 5: Conics: Plane and Cone [4]

The mere use of the term "conic" for ellipse, hyperbola and parabola illustrates perfectly how these curves arise: namely as an intersection of a cone and a plane. In the given example the sectional curve is an ellipse, but by changing the position of the plane hyperbolas or parabolas can be created just as well.

This applet is intended to provide students with an opportunity to discover experimentally the conditions under which the intersection of a plane and a cone form a certain curve.

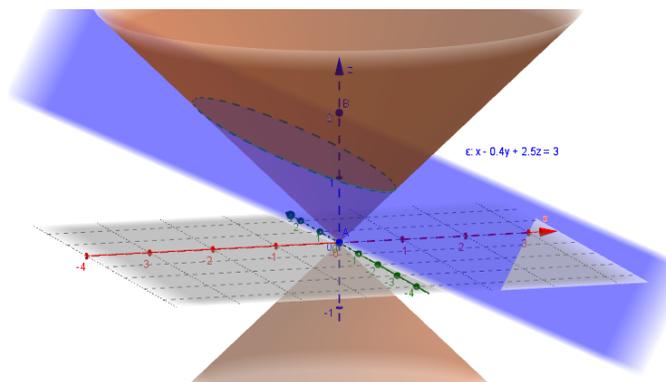
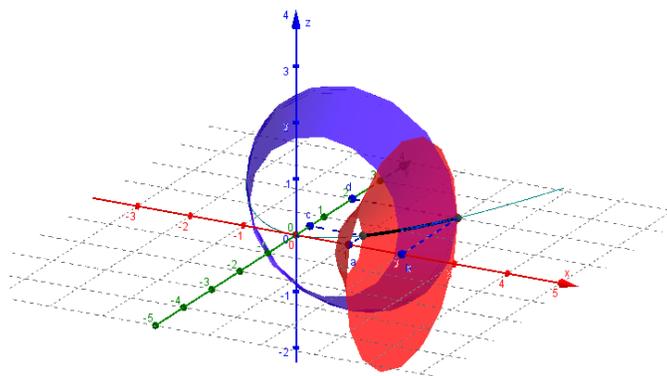


FIGURE 11. Intersection of cone and plane

### 3. CALCULUS

#### Calculus 1: Solids of Revolution [5]

Calculus is a central topic in teaching mathematics in almost all secondary schools. In addition to basics, the calculation of volumes which arise by rotating a graph of a function around the  $x$ - or  $y$ -axis will subsequently be treated as an application of integral calculus. The applet is primarily to help illustrate the process of rotation. This is accomplished by two sliders for the rotation around the  $x$ -axis and the  $y$ -axis. Via an input field the user can enter the desired function whose graph performs the rotation and change the limits as needed.

FIGURE 12. Rotation of a graph around the  $x$ - and  $y$ -axis

#### Calculus 2, 3: Tangents of an Area [6], Tangent Plane of an Area [7]

These two examples demonstrate the visualization of tangents and the tangent plane of a surface. The tangents are formed in  $x$ - and  $y$ -direction by using the partial derivatives with respect to  $x$  and  $y$ .

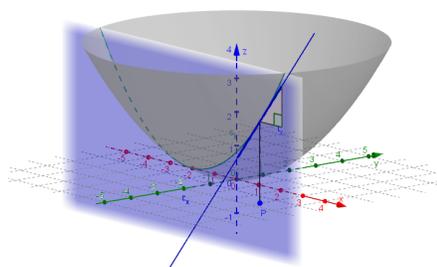


FIGURE 13. Tangents of an area

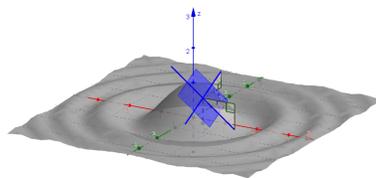


FIGURE 14. Tangent plane

By moving point  $P$ , in which the tangents are formed, their properties can be examined for different functions. The triangles of elevation of the tangents in  $x$ - and  $y$ -direction should also be helpful for investigating the properties of the function. The location of the tangents can be particularly well seen in the horizontal and the vertical projection.

#### 4. GEOMETRY AND CALCULUS

##### Geometry and Calculus 1 [8]

Optimization problem for computing the minimum surface of a cuboid

The common use of dynamically linked representations in different windows is one of the main strengths of GeoGebra. For instance, both two-dimensional graphics views can be linked to the three-dimensional graphics view and the CAS view as shown in the example presented. This is about the calculation of the minimum surface of a cuboid with a square base and a given volume - i.e., a classical optimization problem.

The applet is to assist the students in handling the tasks in several ways:

- to offer an aid to visualizing the problem statement,
- to give them an opportunity for an experimental solution of the problem,
- to allow them to find the solution of the problem with the help of differential calculus.

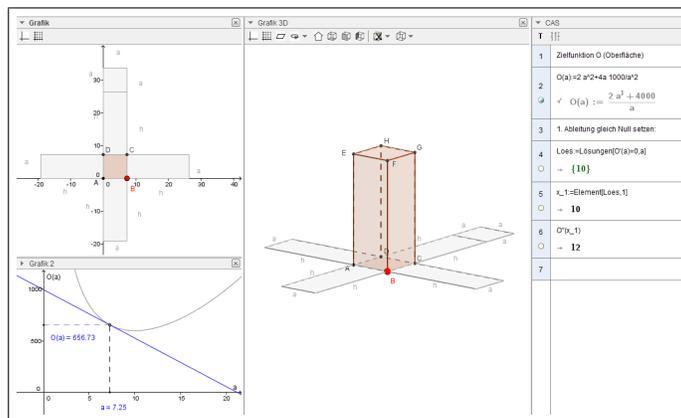


FIGURE 15. Calculation of the minimum surface of a cuboid

An aid towards visualization of the above problem

The graphics view shows the 3D representation of the cube in a three-dimensional coordinate system with a spread net, while the upper graphics view shows the same situation in a horizontal projection. By moving the (red) point B, the dimensions of the cuboid and thus the size of its net change. It is irrelevant whether point B is moved in the two-dimensional or in the three-dimensional graphics view.

This visualization of the real situation appears to be a welcome support for many students for understanding the actual problem.

An alternative for the experimental solution of the problem

Simultaneously with the movement of point  $B$ , a point with coordinates  $(a|O(a))$  is drawn in the second two-dimensional graphics view (Figure 15, bottom left, graphics view 2), where  $a$  is the length of the side edge of the base square and  $O(a)$  indicates the value of the surface of the cuboid as a function of the side edge  $a$ . By moving point  $B$  the length of  $a$  is changed, and thus point  $(a|O(a))$  describes the graph of a function whose minimum value is to be determined. Once you have found the value at which the value of the surface is minimal, it becomes clear that the tangent to the graph is horizontal.

Solution of the problem with the help of differential calculus

In addition to the experimental solution the task can be calculated by means of differential calculus in the CAS of GeoGebra. The setting up of the objective function, the derivation of the function, and the further steps in the conventional treatment of this type of task have been presented here in the form of a finished solution and should be carried out by the students on their own.

## Geometry and Calculus 2 [9]

Optimization problem for the calculation of the maximum volume of a cylinder

*A cylinder is inscribed in a cone with radius  $R$  and height  $H$ . Wanted is a cylinder with maximum volume.*

This example is very similar to the previous and what was said there also applies in this case. However, this task also allows application in the field of argumentation and interpretation of results.

By moving point  $P$  in the Graphics View 3D, the place where the cylinder has the maximum volume can be found approximately. Now if the height  $H$  of the cone is changed by using the slider the position of the maximum for the volume of the cylinder does not change. This can be interpreted in the following way: the radius  $r$  of the cylinder is independent from the height  $H$  of the cone and  $H$  must not be present in the solution.

While in the past the emphasis was on the computational solution by means of differential calculus the approach through dynamic mathematics allows an experimental method for finding the solution and an interpretation of the solution with regard to the parameters of the function. The mathematical solution using the first and second derivatives is carried out with the help of the CAS.

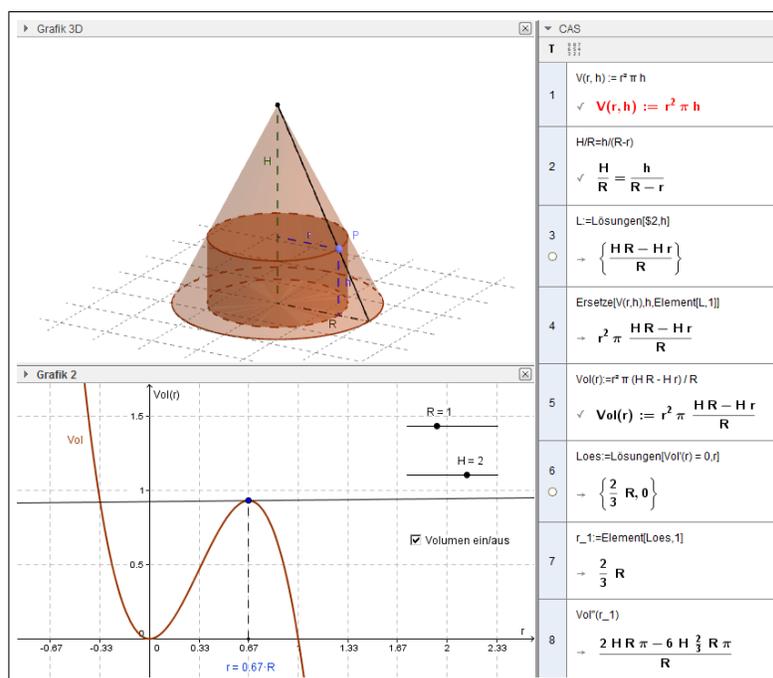


FIGURE 16. Optimization problem for cone and cylinder

## 5. APPLIED MATHEMATICS IN PHYSICS

In conclusion, a few examples demonstrate the meaningful use of GeoGebra applets in science classes. However it is expressly pointed out that applets cannot replace a real experiment, and they are not supposed to do so. But in many cases GeoGebra applets are a useful supplement for the classroom as they facilitate the understanding of emerging circumstances.

### Applied Mathematics in Physics 1

Pendulum and Lissajous-Figures [10]

A pendulum that performs oscillations both in  $x$ - and  $y$ -direction is shown in an animation. The frequency and amplitude of each partial wave can be chosen freely by means of sliders. As a result, the superposition shows a so-called Lissajous figure as seen in figure 17.

Through additional change of the phase shift between the two partial waves students can investigate the behaviour of the pendulum.

### Applied Mathematics in Physics 2

Circular polarized Waves [11]

Each static image cannot totally reflect the essence of a wave, namely their propagation in one direction at a certain speed. It remains up to the viewer to imagine the movement accordingly.

But in an animation, the superposition of two waves oscillating perpendicular to each other and running in the same direction can be shown very easily. Depending on the phase shift the pointer indicating the superposition in a certain place rotates clockwise or counter-clockwise.

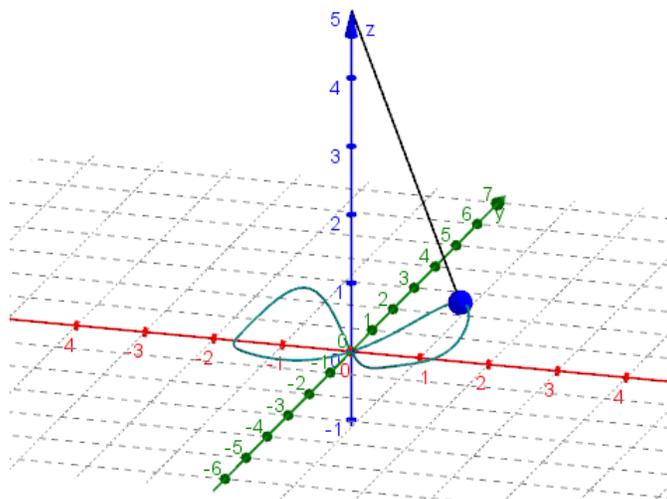


FIGURE 17. Pendulum and Lissajous figure

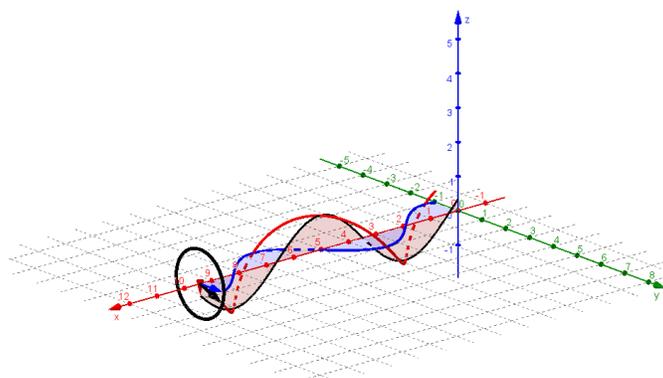


FIGURE 18. Circular polarized waves

### Applied Mathematics in Physics 3

Theory of Relativity: Addition of Velocities [12]

In the special theory of relativity Albert Einstein developed a formula for the addition of two velocities  $v$  and  $u'$  if the movement takes place in a moving reference system. The formula for the addition of two speeds can also be interpreted as a function of two variables so that the graph of this function is a surface in space.

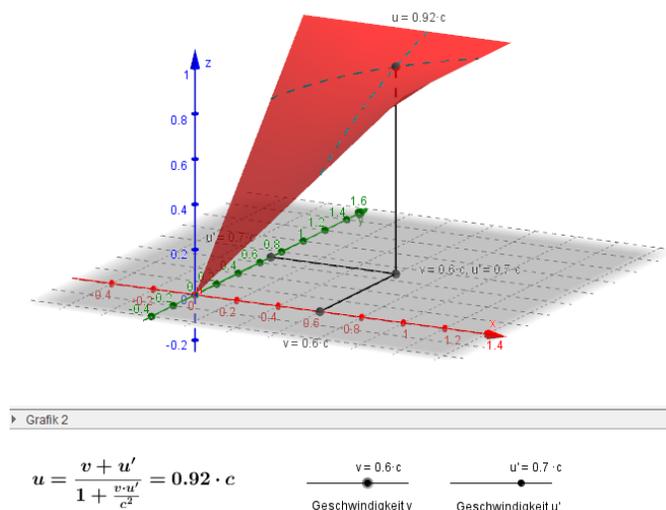


FIGURE 19. Addition of Velocities

As the figure shows, the function value never reaches a value that goes beyond the speed  $c$  of light. The graph is approximately linear for small velocities and shows greater curvature only for greater speeds.

## REFERENCES

- [1] LINDNER, A. Wahre Länge. GeoGebraTube  
<http://www.geogebraTube.org/student/m29132> (Download on Jan 23, 2014)
- [2] LINDNER, A. Kreuzende Geraden. GeoGebraTube  
<http://www.geogebraTube.org/student/m29124> (Download on Jan 23, 2014)
- [3] LINDNER, A. Pyramide 1. GeoGebraTube  
<http://www.geogebraTube.org/student/m29079> (Download on Jan 23, 2014)
- [4] LINDNER, A. Schnitt Kegel - Ebene. GeoGebraTube  
<http://www.geogebraTube.org/student/m30207> (Download on Jan 23, 2014)
- [5] LINDNER, A. Rotationskörper. GeoGebraTube  
<http://www.geogebraTube.org/student/m29080> (Download on Jan 23, 2014)
- [6] LINDNER, A. Tangenten an eine Fläche. GeoGebraTube  
<http://www.geogebraTube.org/student/m43811> (Download on Jan 23, 2014)
- [7] LINDNER, A. Tangentialebene an eine Fläche. GeoGebraTube  
<http://www.geogebraTube.org/student/m37398> (Download on Jan 23, 2014)
- [8] LINDNER, A. Saftbox. GeoGebraTube  
<http://www.geogebraTube.org/student/m30812> (Download on Jan 23, 2014)
- [9] LINDNER, A. Kegel mit eingeschriebenem Zylinder. GeoGebraTube  
<http://www.geogebraTube.org/student/m33895> (Download on Jan 23, 2014)
- [10] LINDNER, A. Schwingendes Pendel und Lissajous-Figuren. GeoGebraTube  
<http://www.geogebraTube.org/student/m32104> (Download on Jan 23, 2014)
- [11] LINDNER, A. Zirkular polarisierte Wellen 3. GeoGebraTube  
<http://www.geogebraTube.org/student/m30599> (Download on Jan 23, 2014)
- [12] LINDNER, A. Addition von Geschwindigkeiten. GeoGebraTube  
<http://www.geogebraTube.org/student/m49657> (Download on Jan 23, 2014)