South Bohemia Mathematical Letters
Volume 22, (2014), No. 1, 26–37.

THE SIMSON-WALLACE LOCUS IN PLANE AND SPACE

PAVEL PECH[1], EMIL SKŘÍŠOVSKÝ[2]

Abstract. In the paper several theorems related to the well-known Simson–Wallace theorem are given. Some properties of the nine-point circle and circumcircle of a given triangle are investigated.
Further the relation between two Simson lines is studied, obtaining Half Angle Theorem. Special attention is paid to Steiner Deltoid curve as the envelope of the system of Simson-Wallace lines whose equation was derived.
Simultaneously the generalization of the theorem into space is described and further examined.

1. Introduction

The Simson–Wallace theorem describes an interesting property regarding the points on the circumcircle of the triangle [1, 3, 4, 9].

Theorem 1.1 (Simson-Wallace)**.** *Let ABC be a triangle and P a point in plane. The feet of the perpendicular lines onto the sides of the triangle are collinear if and only if P lies on the circumcircle of ABC, Fig. 1.*

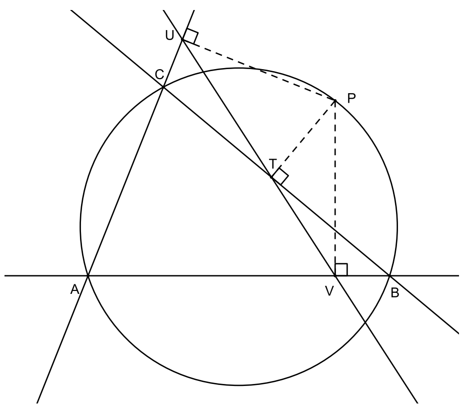

Figure 1. The Simson-Wallace Theorem – points T, U, V are collinear.

Proof. To prove the theorem, we use analytical methods. We adopt a coordinate system where $A = [0, 0]$, $B = [b, 0]$ and $C = [c_1, c_2]$, Fig. 2. We denote the three

Received by the editors .
1991 *Mathematics Subject Classification.*
Key words and phrases. Simson-Walace theorem, family of Simson lines, Steiner deltoid, Cayley's cubic.

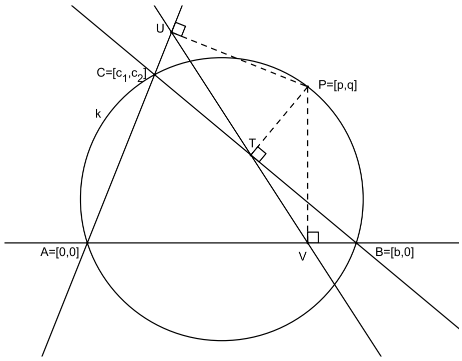

FIGURE 2. Introduction of a coordinate system.

feet to be T, U, V and a point on the circumcircle $P = [p, q]$. We obtain $V = [p, 0]$,

$$U = \left[\frac{c_1(pc_1 + qc_2)}{c_1^2 + c_2^2}, \frac{c_2(pc_1 + qc_2)}{c_1^2 + c_2^2}\right] \quad \text{and}$$

$$T = \left[\frac{p(c_1 - b)^2 + bc_2^2 + qc_2(c_1 - b)}{(c_1 - b)^2 + c_2^2}, \frac{pc_2(c_1 - b) + qc_2^2 - bc_2(c_1 - b)}{(c_1 - b)^2 + c_2^2}\right].$$

To explore the collinearity of these three points, a following determinant describes such a condition

$$\begin{vmatrix} t_1 & t_2 & 1 \\ u_1 & u_2 & 1 \\ v_1 & v_2 & 1 \end{vmatrix} = 0. \tag{1.1}$$

Solving this equation, we get

$$c_2p^2 + c_2q^2 - bc_2p + bc_1q - qc_1^2 - qc_2^2 = 0, \tag{1.2}$$

which is satisfied when P lies on the circumcircle.[1] The equation of the Simson line might be desired and used later:

$$x(pc_1 + qc_2) + y(pc_2 - qc_1) - p(pc_1 + qc_2) = 0. \tag{1.3}$$

□

If we set the determinant (1.1) to be non-zero, then it equals to the twice oriented area of $\triangle TUV$ [10]. J.D. Gergonne showed that the following theorem is true [8].

Theorem 1.2 (Gergonne)**.** *Let ABC be a triangle and P a point in plane. If P lies on a circle concentric to the circumcircle, then the area of the pedal triangle TUV is constant, Fig. 3.*

Remark 1.3. Let us denote, that the converse is not fully true. Given that for a certain value of non-oriented area we have two values of oriented area (one is positive, the other one negative), then the locus might be not only one, but even two circles concentric with the circumcircle.

Corollary 1.4. *The pedal triangle degenerates into the line if and only if P lies on the circumcircle.*

[1]Obviously, coefficients before quadratic terms in (1.2) are non-zero and equal. Substitute into (1.2) to see that for A, B, C it holds, thus describing a circle passing through all vertices.

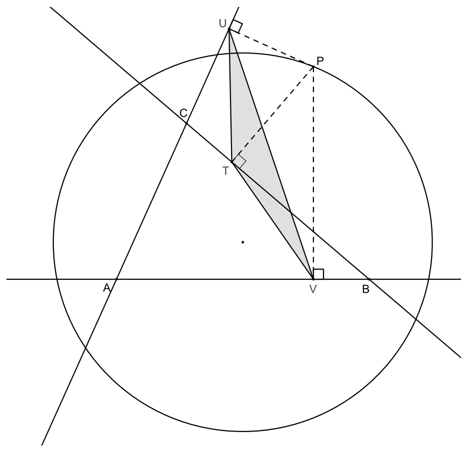

FIGURE 3. Gergonne's generalization – area of TUV is constant.

Remark 1.5. Let us denote, that another known generalization was found by M. de Guzmán [5], who proved that when projecting the point in arbitrary directions, the locus of points for which is the area of so formed triangle constant, is in a general case a conic section, having many interesting properties to be mentioned later.

2. Family of the Simson lines & Steiner Deltoid

Following our study of the Simson-Wallace theorem, we see that position of the respective Simson line is dependent on the position of the point P on the circumcircle. The binding of P on the circumcircle raises a question how the position of s changes when we move P on the circle. Using dynamic geometry software, Fig. 4,

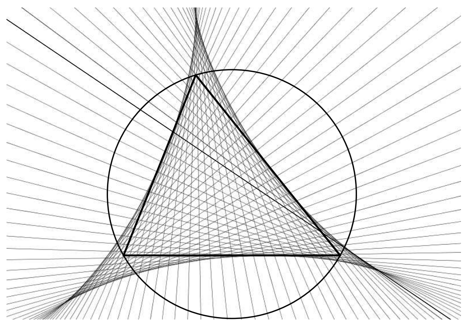

FIGURE 4. A family of Simson lines.

we see that the envelope, whose equation we would like to derive, apparently exists.

We won't discuss all the Simson lines right now, but let us explore the relation between just two Simson lines.

Let's assume two Simson lines s, s' and related points P, P' on the circumcircle oab ABC, respectively. It is obvious that P' is an image of P in rotation around angle θ around the center of circumcircle. Analytically this is the transformation composed of a translation of the center of the circumcircle into the origin, following by a rotation around an arbitrary angle θ and then translating back. The corresponding transformation is given by matrix

$$(2.1)\qquad \begin{bmatrix} \cos\theta & -\sin\theta & -\frac{b\cos\theta}{2}+\frac{\left({c_1}^2+{c_2}^2-bc_1\right)\sin\theta}{2c_2}+\frac{b}{2} \\ \sin\theta & \cos\theta & -\frac{b\sin\theta}{2}-\frac{\left({c_1}^2+{c_2}^2-bc_1\right)\cos\theta}{2c_2}+\frac{{c_1}^2+{c_2}^2-bc_1}{2c_2} \\ 0 & 0 & 1 \end{bmatrix},$$

from which the coordinates of P' might be expressed.[2]

For the angle of lines s and s', following is true: $\cos\varphi = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|}$. Using perpendicular vectors instead, simplifying with respect to (1.2) we get

$$(2.2)\qquad \cos\varphi = \frac{1}{2}\sqrt{(2+2\cos\theta)} = \cos\left(\frac{\theta}{2}\right).$$

Theorem 2.1. *Given a triangle ABC, if P and P' are points on the circumcircle of ABC and $\left|\angle PP'\right| = \theta$, then angle between Simson lines respective to these points is of a half size $\frac{\theta}{2}$; $|\angle s, s'| = \frac{\theta}{2}$ (as shown in Fig. 5).*

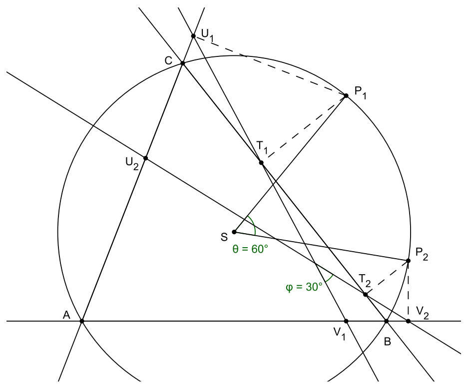

Figure 5. Half Angle Theorem – $\varphi = \frac{\theta}{2}$.

Special case of this theorem is the case, where the angle $\theta = \pi$ and the points P and P' are opposite. Then with respect to Theorem 2.1, their respective Simson lines are perpendicular.

In this case, let us determine at which point they intersect each other. The Simson line s is described by (1.3), to obtain the equation of s', we substitute the coordinates of the opposite point $P' = [b-p; \frac{{c_1}^2+{c_2}^2-bc_1-qc_2}{c_2}]$ into (1.3). Both these lines are dependent on parameters p, q, thus we eliminate them and substitute into (1.2). We obtain

$$(2.3)\qquad {c_1}^2y - bc_1y + bc_1c_2 - 2\,c_1c_2x + 2\,c_2y^2 - {c_2}^2y + 2\,c_2x^2 - bc_2x = 0.$$

We see that it is the nine-point circle, see [9].

[2]The reader will forgive us not expressing the coordinates of P' and line s' explicitly. Because of their complexity, expressing them would be redundant.

Theorem 2.2. *Given a triangle ABC, P and P' points on the circumcircle. If P and P' lie diametrically opposite, then their respective Simson lines are perpendicular and intersect in the nine-point circle, Fig. 6.*

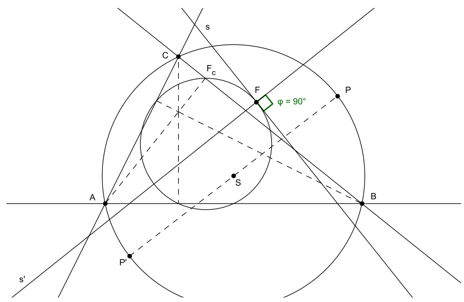

FIGURE 6. Simson lines respective to opposite points are perpendicular and intersect at nine-point circle.

Remark 2.3. While leaving this special case of the Theorem 2.1, let us remark, that in a general case, the locus of intersection points of s and s' is a hypocycloid centered in the center of the nine-point circle. This might be proved in a similar way.

3. Steiner Deltoid

As the angle θ in the Theorem 2.1 approaches zero, the two lines s and s' become the same and the locus of intersections becomes the envelope of the family of Simson lines. Let us remind the definition of an envelope [10]:

Definition 3.1 (Envelope)**.** A curve is called an envelope of a family of curves, if it touches every curve of the family and, conversely, if every point of the curve is the point of contact with any curve of this family.

Parametrizing (1.3) with a single parameter t ($p = t$) we obtain

$$p = t, \qquad q = \frac{t\,(c_1x + c_2y - tc_1)}{tc_2 - c_2x + c_1y} \tag{3.1}$$

and substituting into the condition (1.2) that P lies on circumcircle, we get

$$\begin{aligned} &y\,(t-x)\,{c_2}^2 + \left(-t^3 + (2\,x + b - c_1)\,t^2 - \left(x^2 + 2\,(b-c_1)\,x + y^2\right)\,t + \right.\\ &\qquad \left. + (b-c_1)\,x^2 + y^2c_1\right)c_2 + c_1y\,(t-x)\,(b-c_1) = 0. \end{aligned} \tag{3.2}$$

The envelope satisfies both (see [10])

$$F\,(x,y,t) = 0 \quad \text{and} \quad \frac{\partial F\,(x,y,t)}{\partial t} = 0, \tag{3.3}$$

thus derivation of (3.2) with respect to t gives

$$\left(2\,(x-t)\,c_1 - 3t^2 + 2\,(b+2x)\,t - 2\,bx - x^2 - y^2\right)c_2 + c_1y\,(b-c_1) + {c_2}^2y = 0. \tag{3.4}$$

This is a quadratic equation in t, thus having 2 solutions.[3] Substituting each one in (3.2), we eliminate the parameter t and get two equations.[4] Apparently, each one describes a part of the envelope. To describe the whole, we have to multiply them. After several thorough steps, we get the equation

$$\begin{aligned}
&4\,{c_2}^3\left(x^2+y^2\right)^2-4\,{c_2}^3x^3\,(3\,b-2\,c_1)+20\,{c_2}^2x^2y\left({c_2}^2+bc_1-{c_1}^2\right)+\\
&+4\,{c_2}^3xy^2\,(5\,b-14\,c_1)-12\,{c_2}^2y^3\left({c_2}^2+bc_1-{c_1}^2\right)-c_2\left({c_1}^4-2\,b{c_1}^3+\right.\\
&\left.+\left(b^2-2\,{c_2}^2\right){c_1}^2+14\,bc_1{c_2}^2-12\,b^2{c_2}^2+{c_2}^4\right)x^2-(22\,b-4\,c_1)\\
&\left({c_2}^2+bc_1-{c_1}^2\right){c_2}^2xy+c_2\left(12\,{c_2}^4-(b+2\,c_1)\,(b-10\,c_1)\,{c_2}^2+12\,{c_1}^2\right.\\
&\left.(b-c_1)^2\;\;y^2+2\,c_2\,(b-c_1)\left({c_1}^4-2\,b{c_1}^3+\left(2\,{c_2}^2+b^2\right){c_1}^2+{c_2}^4-\right.\right.\\
&\left.-2\,b^2{c_2}^2\right)x-2\,\left({c_2}^2+bc_1-{c_1}^2\right)\left(2{c_1}^4-4\,b{c_1}^3+2\,\left(2\,{c_2}^2+b^2\right){c_1}^2-\right.\\
&\left.-3\,bc_1{c_2}^2+2{c_2}^4-b^2{c_2}^2\right)y-c_2\,(b-c_1)^2\left({c_1}^2-bc_1+{c_2}^2\right)^2=0,
\end{aligned}\tag{3.5}$$

which is a curve of a fourth degree, Fig. 7. It was first described by J. Steiner in

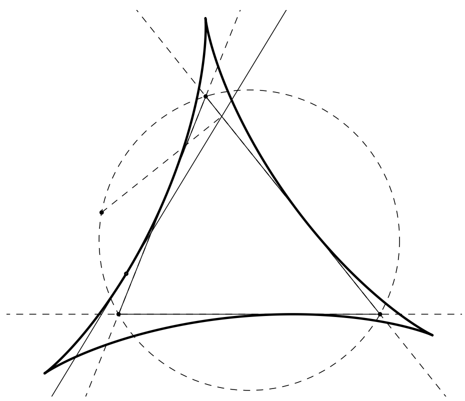

FIGURE 7. Steiner Deltoid as the envelope of Simson lines.

1857. Let us state the following theorem:

Theorem 3.2 (Steiner Deltoid)**.** *The envelope of Simson lines is a curve called Steiner Deltoid, given by the equation* (3.5)*, Fig. 7.*

There are several interesting properties regarding Steiner Deltoid, that are mentioned below.

Theorem 3.3. *Steiner Deltoid is tangent to the sides of ABC. These points are symmetrical with the feet of altitudes with respect to the midpoints, Fig. 8.*

Proof. One of the solutions of (3.5) and $y = 0$ is a point $P = [b - c_1; 0]$. The other two lie outside of the side AB of the triangle. □

[3]Since p is an x-coordinate on the circumcircle, we get two points having the same x-coordinate.
[4]We skip again for the same reason explicit expressions.

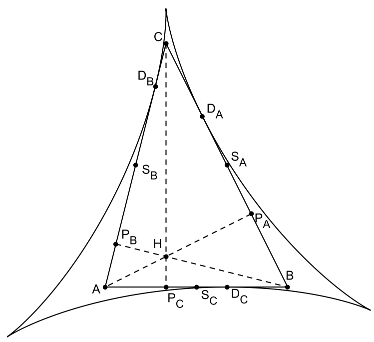

FIGURE 8. Steiner Deltoid touches triangle ABC in points symmetric to feet of altitudes.

The distance between them and the midpoint is $R = \frac{\sqrt{({c_1}^2+{c_2}^2)\left((c_1-b)^2+{c_2}^2\right)}}{2\,c_2}$, which is twice the radius of the nine-point circle and the same as radius of circumcircle of ABC (see (1.2)). Hence:

Theorem 3.4. *Circle centered in the midpoint, passing through the intersections of the Deltoid outside of AB, is tangent to the nine-point circle and to a circle concentric with the nine-point circle, with three times the radius, Fig. 9.*

Corollary 3.5. *Due to the symmetry, the point of intersection of that circle and the nine-point circle lies in the middle of segment CH, Fig. 9.*

Corollary 3.6. *The radius of the nine-point circle is of a half of the circumcircle.*

Similar theorem, regarding the circle of the same radius, might be stated for

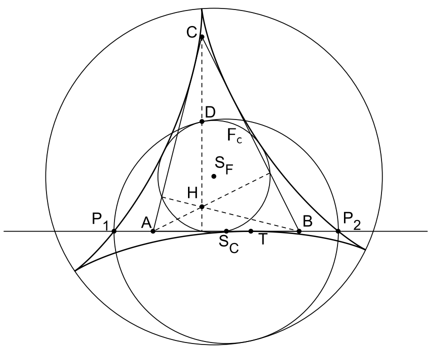

FIGURE 9. Two intersections of Steiner Deltoid and the side of the triangle are symmetric with respect to midpoint; circle passing through them is tangent to the nine-point circle in midpoint of CH.

every point on the nine-point circle. All these circles would have as their envelope the greater circle – in which is the Deltoid curve inscribed. These properties result from the fact, that Deltoid is a hypocycloid, a curve generated by tracing a point on a circle rolling within a circle of three times the radius. Hence:

Corollary 3.7. *Steiner Deltoid is a curve generated by tracing a point on the circle k rolling outside of the nine-point circle (or within the greater one).*

Given that all these circles are tangent to nine-point-circle, a point exists, in which the Steiner Deltoid touches the nine-point circle. Thus:

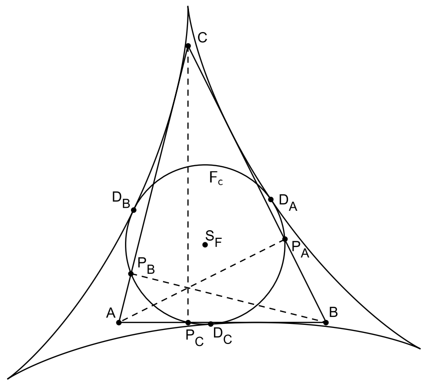

FIGURE 10. Steiner Deltoid touches the nine-point circle.

Theorem 3.8. *The Steiner Deltoid is tangent to the nine-point circle, Fig. 10.*

Now, we'll study the properties resulting from the fact that the Deltoid is an envelope of Simson lines.

Theorem 3.9. *The Deltoid is tangent to the altitudes of a triangle.*

Proof. A Simson line respective to the vertex of ABC is the altitude. Deltoid as an envelope is tangent to this line by definition. ☐

Regarding the symmetry of Deltoid, center of the nine-point circle is center of the Deltoid. Simson line passing through this point is its axis of symmetry. There

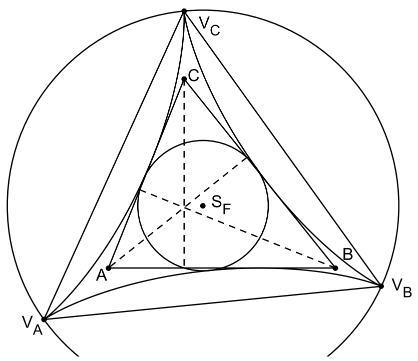

FIGURE 11. The three cusps (singular points) of Steiner Deltoid.

are three lines, each passing through the cusps of Deltoid.

These cusps are obviously singular points of the curve, thus their partial derivatives are equal to zero. From derivation of (3.5) we see that it leads on a cubic equation, making the cusps to be non-constructible using Euclidean geometry. The condition for Simson line passing through center S_F leads to the cubic expression as well.

Theorem 3.10. *The Deltoid has three cusps, Fig. 11.*

At last let us remark that the cusps form an equilateral triangle, with sides parallel to the Morley's triangle [6], Fig. 12.

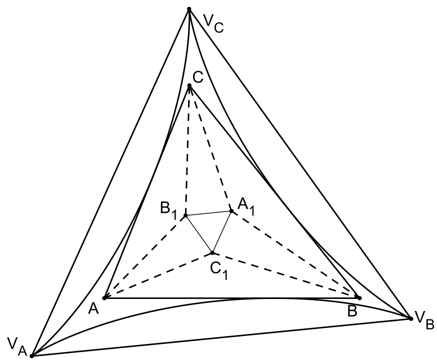

FIGURE 12. The cusps form an equilateral triangle with sides parallel to these of Morley triangle.

4. Generalization in 3D Euclidean space

An obvious way to generalize the Simson-Wallace theorem is to generalize it into 3D. Given a tetrahedron, we explore the locus of points P, for which the four points T, U, V, W (intersections of the perpendicular lines onto the faces with the faces itself) are coplanar.

Adopting a cordinate system in E_3, where $A = [0,0,0]$, $B = [b,0,0]$, $C = [c_1,c_2,0]$, $D = [d_1,d_2,d_3]$ and $P = [p_1,p_2,p_3]$, using similar methods as in the plane, we explore the following determinant

$$\begin{vmatrix} t_1 & t_2 & t_3 & 1 \\ u_1 & u_2 & u_3 & 1 \\ v_1 & v_2 & v_3 & 1 \\ w_1 & w_2 & w_3 & 1 \end{vmatrix} = 0, \tag{4.1}$$

which describes the condition for T, U, V and W to be coplanar. Solving this equation, we get the third degree expression, thus getting the locus of the points to be a cubic surface

$$
\begin{aligned}
&{c_2}^2{d_3}^2x^2y - c_2{d_3}^2\left(2\,c_1 - b\right)xy^2 + {d_3}^2c_1\left(b - c_1\right)y^3 + c_2d_3\left({d_2}^2 + {d_3}^2 - c_2d_2\right)x^2z + \\
&+ 2\,d_3d_2c_2\left(c_1 - d_1\right)xyz + d_3\left[\left({d_1}^2 - d_1b + {d_3}^2\right)c_2 + c_1d_2\left(b - c_1\right)\right]zy^2 + \\
&+ c_2{d_3}^2\left(b - 2\,d_1\right)xz^2 + {d_3}^2\left[{c_2}^2 - 2\,c_2d_2 - c_1\left(b - c_1\right)\right]yz^2 + \left[c_2\left({d_2}^2 - d_1\,(b - \right.\right. \\
&\left.\left. - d_1)\right) - {c_2}^2d_2 + c_1d_2\left(b - c_1\right)\right]d_3z^3 - {c_2}^2{d_3}^2bxy + {d_3}^2c_1bc_2y^2 - c_2d_3b\left({d_2}^2 + {d_3}^2 - \right. \\
&\left. - c_2d_2\right)xz - d_3\left[\left({d_2}^2 + {d_3}^2\right){c_1}^2 + \left(2\,d_2\left(b - d_1\right)c_2 - b\left({d_2}^2 + {d_3}^2\right)\right)c_1 + \left({d_3}^2 - \right.\right. \\
&\left.\left. - d_1\left(b - d_1\right)\right){c_2}^2\right]zy + \left[c_2\left(\left(b - 2\,d_1\right)c_1 + d_1b\right){d_2}^2 - c_1\left(b - c_1\right){d_2}^3 + \left(\left({d_1}^2 - \right.\right.\right. \\
&\left.\left.\left. - d_1b + {d_3}^2\right){c_2}^2 - c_1{d_3}^2\left(b - c_1\right)\right)d_2 + d_1c_2{d_3}^2b\right]z^2 = 0.
\end{aligned}
\tag{4.2}
$$

This equation describes the well-known Cayley's cubic nodal surface — a surface with four nodes (conical points, in vertices of the triangle).

It is really interesting, that when generalized into 3D, the locus is not the circumsphere (circumscribed sphere) of the tetrahedron, but the Cayley's cubic.

For example, assuming the tetrahedron to be regular with the edge of a ($b = a$, $c_1 = \frac{a}{2}$, $c_2 = \frac{a\sqrt{3}}{2}$, $d_1 = \frac{a}{2}$, $d_2 = \frac{\sqrt{3}a}{6}$ and $d_3 = \frac{\sqrt{6}a}{3}$, where $a = 4$), the equation (4.2) is simplified into

$$3\sqrt{2}z\left(x^2 + y^2 - 4x\right) + \sqrt{3}\left(10z^2 + 8y^2\right) + 6xy\left(x - 4y\right) + 2\quad y^3 + \sqrt{2}z^3\quad - 4\sqrt{6}yz = 0,$$

which corresponds to Fig. 13. As for other example, for trirectangular tetrahedron

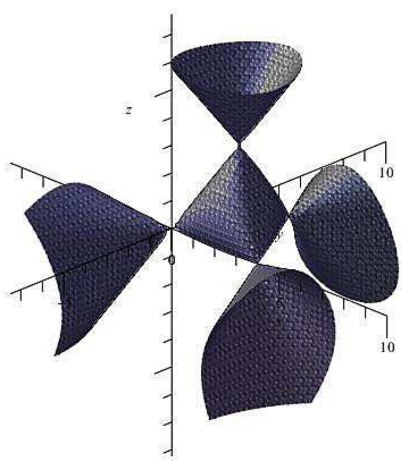

FIGURE 13. Cayley's Cubic as the locus of points (Regular Tetrahedron).

($A = [0, 0, 0]$, $B = [a, 0, 0]$, $C = [0, a, 0]$ and $D = [0, 0, a]$, where $a = 4$) we obtain

$$\left(x^2 + y^2\right)z + \left(x + y\right)\left(z^2 + xy\right) - 4\,xz - 4\,xy - 4\,yz = 0,$$

which is shown in Fig. 14. These results give us the following theorem.

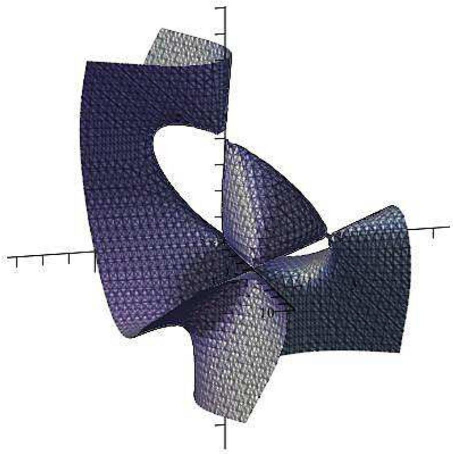

FIGURE 14. Cayley's Cubic as the locus of points (Trirectangular Tetrahedron).

Theorem 4.1. *Given a tetrahedron $ABCD$ in E_3, the locus of points P in space, for which the feet of the perpendicular lines are coplanar, is Cayley's cubic nodal surface given by equation* (4.2).

Theorem 4.2. *Cayley's cubic is passing through the vertices of the tetrahedron.*

Proof. If P is a vertex A, then the three of four feet degenerate into the same point (the vertex A). Two points are always coplanar. The same holds by analogy for the remaining three vertices. Other way to prove that, is to substitute coordinates of vertices A, B, C and D into (4.2). □

Corollary 4.3. *Vertices of Cayley's cubic are nodal points.*

Proof. All first partial derivatives of (4.2) are in $A = [0, 0, 0]$ equal to zero, so the vertex A is a singular point. For other vertices it holds by analogy. □

Theorem 4.4. *All edges of the tetrahedron lie on Cayley's cubic.*

Proof. The same as in proof before, if P lies on the edge of the tetrahedron, then two feet degenerate into P. Three points (P and the other two feet) are always coplanar. An analytical proof is possible as well. □

Remark 4.5. It is possible to generalize the Gergonne theorem into space in a similar way. For the area of the pedal tetrahedron (whose vertices are feet of the perpendiculars onto the faces of the tetrahedron) following equation holds [10],

$$\begin{vmatrix} t_1 & t_2 & t_3 & 1 \\ u_1 & u_2 & u_3 & 1 \\ v_1 & v_2 & v_3 & 1 \\ w_1 & w_2 & w_3 & 1 \end{vmatrix} = 6\, V_{TUVW}. \tag{4.3}$$

Solving this equation, we get a similar expression as in (4.2)

$$V(x, y, z) - 6\, V_{TUVW} \cdot c = 0, \tag{4.4}$$

where c is constant and $V(x, y, z)$ expression described in (4.2). The plotted surface is in Fig. 15. Apparent analogy to Fig. 13 might be observed.

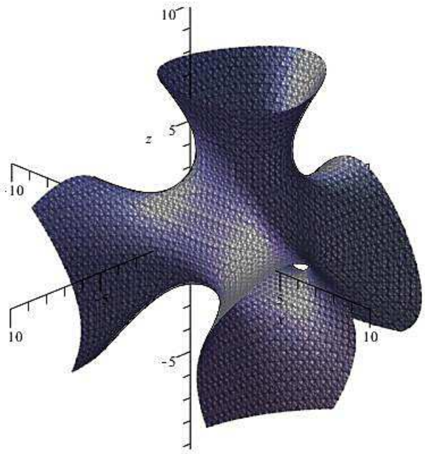

FIGURE 15. Cayley's Cubic as the locus of points, Gergonne generalization, $V_{TUVW} = 1$.

Remark 4.6. As in 2D case, the locus might in fact consist of two surfaces – one for positive value of oriented volume and the other one for negative oriented volume of the tetrahedron. But their non-oriented volume is the same.

References

[1] Baker, H. F.: An introduction to plane geometry: with many examples. Cambridge University Press, Cambridge, 1943.

[2] Bradley, J. C., Smith, C. G.: On a Construction of Hagge. Forum Geometricorum, 7 (2007), 231–247. [online]: http://forumgeom.fau.edu/FG2007volume7/FG200730index.html

[3] Coxeter, H. S. M.: Introduction to Geometry. (2nd ed.). J. Wiley & Sons, New York, 1989.

[4] Coxeter, H. S. M., Greitzer, S. L.: Geometry Revisited. New Mathematical Library, The Mathematical Association of America, Washington, 1967.

[5] de Guzmán M.: An Extension of the Wallace-Simson Theorem: Projecting in Arbitrary Directions. American Mathematical Monthly, 106 (1999), 574–580.

[6] de Guzmán, M.: The envelope of the Wallace-Simson lines of a triangle: a simple proof of the Steiner theorem on the Deltoid. RACSAM, Revista de la Real Academia de Ciencias Exactas, 95 (2001), 57–64.

[7] Lockwood, E. H.: A Book of Curves. Cambridge University Press, 1961.

[8] Pech, P. Klasické vs. počítačové metody při řešení úloh v geometrii. České Budějovice, Jihočeská univerzita, 2005. [Czech]

[9] Pech, P., Skříšovský, E.: On the Simson–Wallace theorem. South Bohemia Mathematical Letters 21 (2013), 59–66.

[10] Rektorys, K.: Survey of Applicable Mathematics. Mass.: M.I.T. Press Cambridge, 1969.

[11] Skříšovský, E.: Simsonova věta a její zobecnění v rovině a prostoru. Original paper from National Competition, 2013. [Czech]

[1]Faculty of Education, University of South Bohemia, České Budějovice [2]Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic.

E-mail address: [1] pech@pf.jcu.cz, [2] emil.skrisovsky@gmail.com