South Bohemia Mathematical Letters
Volume 23, (2015), No. 1, 73-85.

ON POWER POMPEIU'S TYPE INEQUALITIES FOR DOUBLE INTEGRALS

MEHMET ZEKI SARIKAYA, SAMET ERDEN, AND HATICE YALDIZ

ABSTRACT. The main of this paper is to derive some new inequalities of Ostrowski type using Pompeiu's mean value theorem for double integrals involving functions of two independent variables.

INTRODUCTION

In 1938, the classical integral inequality established by Ostrowski [10] as follows:

Theorem 0.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{0.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (0.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some midpoint, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (0.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([3]-[5], [8],[14]-[18]) for integral inequalities in several independent variables.. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

In 1946, Pompeiu [12] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem.*

Theorem 0.2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1\neq x_2$ in $[a,b]$, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1f(x_2)-x_2f(x_1)}{x_1-x_2}=f(\xi)-\xi f'(\xi).$$

In [13], E.C. Popa using a mean value theorem obtained following theorem.

Key words and phrases. Ostrowski's inequality, Pompeiu's inequalities, absolutely continuous function, double integrals.

Theorem 0.3. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that $\alpha\notin[a,b]$. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|\left(\frac{a+b}{2}-\alpha\right)f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f-l_\alpha f'\right\|_\infty .\end{aligned}$$

In [11], the authors have proved the following Ostrowski type inequality:

Theorem 0.4. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for $1\leq p,\ q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}+\frac{1}{b-a}\int_a^b f(t)dt\right|\leq PU(x,p)\left\|f-lf'\right\|_p,$$

for $x\in[a,b]$, where

$$\begin{aligned}PU(x,p)\ \ :&=(b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{p}}\right.\\&\left.+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\right].\end{aligned}$$

In the cases $(p,q)=(1,\infty),(\infty,1)$ and $(2,2)$ the quantity $PU(x,p)$ has to be taken as the limit as $p\longrightarrow1,\infty$ and 2, respectively.

In [6], Dragomir has proved the Ostrowski type inequalities for complex valued absolutely continuous functions as follows:

Theorem 0.5. *Let $f:[a,b]\longrightarrow\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq0$, and and $f'l-rf\in L_\infty[a,b]$, then for any $x\in[a,b]$ we have*

$$\begin{aligned}&\left|\frac{b^{r+1}-a^{r+1}}{r+1}f(x)-x^r\int_a^b f(t)dt\right|\\ \leq\ &\frac{1}{|r|}\left\|f'l-rf\right\|_\infty\\ &\times\begin{cases}\frac{2rx^{r+1}-x^r(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1}, & if\ r>0\\ \frac{x^r(a+b)(r+1)-2rx^{r+1}-b^{r+1}-a^{r+1}}{r+1}, & if\ r\in(-\infty,1)\backslash\{-1\}.\end{cases}\end{aligned}$$

Also, for $r=-1$, we have

$$\left|f(x)\ln\frac{b}{a}-\frac{1}{x}\int_a^b f(t)dt\right|\leq\frac{1}{|r|}\left\|f'l+f\right\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x\in[a,b]$, provided $f'l+f\in L_\infty[a,b]$.

The interested reader is also referred to ([1], [2], [6], [7], [9], [11], [13], [19]-[21]) for integral inequalities by using Pompeiu's mean value theorem. The main aim of this paper is to establish some Pompeiu's type inequlity for complex valued absolutely continuous functions with double integrals involving functions of two independent variables.

1. Main Result

To prove our theorems, we need the following lemma:

Lemma 1.1. *$f : \Delta = [a, b] \times [c, d] \to \mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t, s) \in \Delta$ with $0 < a < b$, $0 < c < d$. If $r \in \mathbb{R}$, $r \neq 0$, then for any $(t, s), (x, y) \in \Delta$ with $x \neq y \neq t \neq s$, we have*

$$x^r y^r s^r t^r \int_t^x \int_s^y \frac{1}{u^{r+1} v^{r+1}} \left[uv f_{uv}(u, v) - ru f_u(u, v) - rv f_v(u, v) + r^2 f(u, v) \right] dv du$$

$$= s^r t^r f(x, y) - y^r t^r f(x, s) - x^r s^r f(t, y) + x^r y^r f(t, s).$$

Proof. Since f is continuously differantiable function, $\frac{f(u,v)}{u^r v^r}$ is an absolutely continuous function on Δ. Then we get

$$\int_t^x \int_s^y \frac{\partial^2}{\partial u \partial v} \left[\frac{f(u, v)}{u^r v^r} \right] dv du = \int_t^x \frac{\partial}{\partial u} \left(\frac{f(u, y)}{u^r y^r} - \frac{f(u, s)}{u^r s^r} \right) du = \frac{f(x, y)}{x^r y^r} - \frac{f(x, s)}{x^r s^r} - \frac{f(t, y)}{t^r y^r} + \frac{f(t, s)}{t^r s^r}, \tag{1.1}$$

for all $(t, s), (x, y) \in \Delta$ with $x \neq y \neq t \neq s$. On the other hand,

$$\frac{\partial^2}{\partial u \partial v} \left[\frac{f(u, v)}{u^r v^r} \right] = \frac{1}{u^{r+1} v^{r+1}} \left[uv f_{uv}(u, v) - ru f_u(u, v) - rv f_v(u, v) + r^2 f(u, v) \right] \tag{1.2}$$

for all $(t, s), (x, y) \in \Delta$ with $x \neq y \neq t \neq s$. Thus, from (1.1) and (1.2), it follows that

$$\int_t^x \int_s^y \frac{\partial^2}{\partial u \partial v} \left[\frac{f(u, v)}{u^r v^r} \right] dv du$$

$$= \int_t^x \int_s^y \frac{1}{u^{r+1} v^{r+1}} \left[uv f_{uv}(u, v) - ru f_u(u, v) - rv f_v(u, v) + r^2 f(u, v) \right] dv du$$

$$= \frac{f(x, y)}{x^r y^r} - \frac{f(x, s)}{x^r s^r} - \frac{f(t, y)}{t^r y^r} + \frac{f(t, s)}{t^r s^r}$$

which this completes the proof. □

Theorem 1.2. *$f : \Delta \to \mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t, s) \in \Delta$ with $0 < a < b$, $0 < c < d$. If $r \in \mathbb{R}$,*

$r \neq 0$, then for any $(t,s),(x,y) \in \Delta$ with $x \neq y \neq t \neq s$, we have

$$\begin{aligned}
&(1.3)\\
&\left|s^r t^r f(x,y) - y^r t^r f(x,s) - x^r s^r f(t,y) + x^r y^r f(t,s)\right| \\
\leq\ & x^r y^r s^r t^r \begin{cases}
\frac{1}{r^2}\left\|l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f\right\|_\infty \left|\frac{1}{x^r}-\frac{1}{t^r}\right|\left|\frac{1}{y^r}-\frac{1}{s^r}\right| \\
\qquad \text{if } l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \in L_\infty(\Delta) \\
\begin{cases}
\left\|l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f\right\|_p \\ \times \frac{\left|x^{1-(r+1)q}-t^{1-(r+1)q}\right|^{\frac{1}{q}}\left|y^{1-(r+1)q}-s^{1-(r+1)q}\right|^{\frac{1}{q}}}{(1-q(r+1))^{\frac{2}{q}}} & \text{if } r \neq -\frac{1}{p} \\
\left\|l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f\right\|_p \\ \times \left|\ln x - \ln t\right|\left|\ln y - \ln s\right| & \text{if } r = -\frac{1}{p}
\end{cases} \\
\qquad \text{if } l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \in L_p(\Delta) \\
\left\|l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f\right\|_1 \frac{1}{\min\{t^{r+1},x^{r+1}\}}\frac{1}{\min\{s^{r+1},y^{r+1}\}} \\
\qquad \text{if } l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \in L_1(\Delta)
\end{cases}
\end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $l_1(x,y)=xy$, $l_2(x,\cdot)=x$ and $l_3(\cdot,y)=y$ for all $(x,y)\in\Delta$.

Proof. From Lemma 1.1 and by using modulus, we have

$$\begin{aligned}
&(1.4)\\
&\left|s^r t^r f(x,y) - y^r t^r f(x,s) - x^r s^r f(t,y) + x^r y^r f(t,s)\right| \\
=\ & x^r y^r s^r t^r \left|\int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}}\left[uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v)\right] dv du\right| \\
\leq\ & x^r y^r s^r t^r \left|\int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}}\left|uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v)\right| dv du\right|.
\end{aligned}$$

Firstly, we will consider the case $p=\infty$ and $q=1$. Then, we have

$$\begin{aligned}
&\left|\int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}}\left|uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v)\right| dv du\right| \\
\leq\ & \sup_{(u,v)\in[t,x]\times[s,y]}\left|uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v)\right| \left|\int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}} dv du\right| \\
=\ & \frac{1}{r^2}\left\|l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f\right\|_\infty \left|\frac{1}{x^r}-\frac{1}{t^r}\right|\left|\frac{1}{y^r}-\frac{1}{s^r}\right|.
\end{aligned}$$

Now, consider the case $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, by using Hölder's inequality we have

$$\begin{aligned}
&\left| \int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}} \left| uvf_{uv}(u,v) - ruf_u(u,v) - rvf_v(u,v) + r^2 f(u,v) \right| dvdu \right| \\
\leq & \left| \int_t^x \int_s^y \left| uvf_{uv}(u,v) - ruf_u(u,v) - rvf_v(u,v) + r^2 f(u,v) \right|^p dvdu \right|^{\frac{1}{p}} \left| \int_t^x \int_s^y \left(\frac{1}{u^{r+1}v^{r+1}} \right)^q dvdu \right|^{\frac{1}{q}} \\
= & \begin{cases} \left\| l_1 f_{uv} - rl_2 f_u - rl_3 f_v + r^2 f \right\|_p \frac{\left| x^{1-(r+1)q} - t^{1-(r+1)q} \right|^{\frac{1}{q}} \left| y^{1-(r+1)q} - s^{1-(r+1)q} \right|^{\frac{1}{q}}}{(1-q(r+1))^{\frac{2}{q}}} & \text{if } r \neq -\frac{1}{p} \\ \left\| l_1 f_{uv} - rl_2 f_u - rl_3 f_v + r^2 f \right\|_p \left| \ln x - \ln t \right| \left| \ln y - \ln s \right| & \text{if } r = -\frac{1}{p} \end{cases}
\end{aligned}$$

Finally, we consider the case $p = 1$ and $q = \infty$. Then, we get

$$\begin{aligned}
&\left| \int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}} \left| uvf_{uv}(u,v) - ruf_u(u,v) - rvf_v(u,v) + r^2 f(u,v) \right| dvdu \right| \\
\leq & \left| \int_t^x \int_s^y \left| uvf_{uv}(u,v) - ruf_u(u,v) - rvf_v(u,v) + r^2 f(u,v) \right| dvdu \right| \sup_{(u,v)\in[t,x]\times[s,y]} \left(\frac{1}{u^{r+1}v^{r+1}} \right) \\
= & \left\| l_1 f_{uv} - rl_2 f_u - rl_3 f_v + r^2 f \right\|_1 \frac{1}{\min\{t^{r+1}, x^{r+1}\}} \frac{1}{\min\{s^{r+1}, y^{r+1}\}}.
\end{aligned}$$

This completes the proof. □

example 1.3. *Let us consider function f defined as $f(u,v) = u^2v^2$. Then we have*

$$f_u(u,v) = 2uv^2, \ f_v(u,v) = 2vu^2 \text{ and } f_{uv}(u,v) = 4uv.$$

If we take $a = 2$, $b = 5$, $c = 1$, $d = 6$ and $r = 1$, we get

$$(1.5) \qquad |y-s|\,|x-t| \leq \begin{cases} \left\| u^2v^2 \right\|_\infty \frac{|y-s||x-t|}{xyst}, & \text{for } p = \infty \\ \frac{\sqrt{|x^5-t^5||y^5-s^5|}}{5} \frac{\sqrt{|y^{-3}-s^{-3}||x^{-3}-t^{-3}|}}{3}, & \text{for } p = 2 \\ \frac{|y^3-s^3||x^3-t^3|}{9} \frac{1}{\min\{t^2,x^2\}} \frac{1}{\min\{s^2,y^2\}}, & \text{for } p = 1. \end{cases}$$

In particular, if we take $r = \frac{-1}{2}$ for $p = 2$, we deduce

$$(1.6) \qquad |y-s|\,|x-t| \leq \frac{25}{4} \frac{\sqrt{|x^5-t^5|\,|y^5-s^5|}}{5} \left| \ln x - \ln t \right| \left| \ln y - \ln s \right|.$$

If we chose $t = 3$, $x = 4$, $s = 2$ and $y = 5$ in (1.5) and (1.6), we obtain

$$3 \leq \begin{cases} 22.5, & \text{for } p = \infty \\ 17.25, & \text{for } p = 2 \\ 19.9, & \text{for } p = 1 \end{cases}$$

and

$$3 \leq 510.75.$$

Theorem 1.4. *$f:\Delta\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ with $0<a<b,\ 0<c<d$. If $r\in\mathbb{R}$, $r\neq 0$ and $l_1f_{uv}-rl_2f_u-rl_3f_v+r^2f\in L_\infty(\Delta)$, then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$\begin{aligned}(1.7)\qquad &\left|\frac{\left(b^{r+1}-a^{r+1}\right)\left(d^{r+1}-c^{r+1}\right)}{(r+1)^2}f(x,y)-\frac{b^{r+1}-a^{r+1}}{r+1}y^r\int_c^d f(x,s)ds\right.\\ &\left.-\frac{d^{r+1}-c^{r+1}}{r+1}x^r\int_a^b f(t,y)dt+x^ry^r\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ &\frac{1}{r^2}\left\|l_1f_{uv}-rl_2f_u-rl_3f_v+r^2f\right\|_\infty\\ &\times\left[\frac{2rx^{r+1}-(a+b)(r+1)x^r+\left(a^{r+1}+b^{r+1}\right)}{r+1}\right]\\ &\times\left[\frac{2ry^{r+1}-(c+d)(r+1)y^r+\left(c^{r+1}+d^{r+1}\right)}{r+1}\right]\end{aligned}$$

for $r\neq -1$. Also, for $r=-1$, we have

$$\begin{aligned}(1.8)\qquad &\left|\left(\ln\frac{b}{a}\right)\left(\ln\frac{d}{c}\right)f(x,y)-y^{-1}\left(\ln\frac{b}{a}\right)\int_c^d f(x,s)ds\right.\\ &\left.-x^{-1}\left(\ln\frac{d}{c}\right)\int_a^b f(t,y)dt+x^{-1}y^{-1}\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ &4\left\|l_1f_{uv}+l_2f_u+l_3f_v+f\right\|_\infty\\ &\times\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right)\left(\ln\frac{y}{\sqrt{cd}}+\frac{\frac{c+d}{2}-y}{y}\right)\end{aligned}$$

for $(x,y)\in\Delta$, provided $l_1f_{uv}+l_2f_u+l_3f_v+f\in L_\infty(\Delta)$. The constant 4 is best possible in (1.8).

Proof. We observe that

$$\begin{aligned}&\int_a^b\int_c^d\left[s^rt^rf(x,y)-y^rt^rf(x,s)-x^rs^rf(t,y)+x^ry^rf(t,s)\right]dsdt\\ =\ &\frac{\left(b^{r+1}-a^{r+1}\right)\left(d^{r+1}-c^{r+1}\right)}{(r+1)^2}f(x,y)-\frac{b^{r+1}-a^{r+1}}{r+1}y^r\int_c^d f(x,s)ds\\ &-\frac{d^{r+1}-c^{r+1}}{r+1}x^r\int_a^b f(t,y)dt+x^ry^r\int_a^b\int_c^d f(t,s)dsdt.\end{aligned}$$

By using the first inequality in (1.3) for $r \neq -1$, it follows that

$$\begin{aligned}
(1.9)\qquad & \left| \frac{\left(b^{r+1}-a^{r+1}\right)\left(d^{r+1}-c^{r+1}\right)}{(r+1)^2} f(x,y) - \frac{b^{r+1}-a^{r+1}}{r+1} y^r \int_c^d f(x,s)ds \right. \\
& \left. - \frac{d^{r+1}-c^{r+1}}{r+1} x^r \int_a^b f(t,y)dt + x^r y^r \int_a^b \int_c^d f(t,s)dsdt \right| \\
\leq & \int_a^b \int_c^d \left| s^r t^r f(x,y) - y^r t^r f(x,s) - x^r s^r f(t,y) + x^r y^r f(t,s) \right| dsdt \\
\leq & \int_a^b \int_c^d \frac{x^r y^r s^r t^r}{r^2} \left\| l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \right\|_\infty \left| \frac{1}{x^r} - \frac{1}{t^r} \right| \left| \frac{1}{y^r} - \frac{1}{s^r} \right| dsdt \\
= & \frac{1}{r^2} \left\| l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \right\|_\infty \left[\int_a^b |t^r - x^r| \, dt \right] \left[\int_c^d |s^r - y^r| \, ds \right].
\end{aligned}$$

Then for $r > 0$, we have

$$\begin{aligned}
\int_a^b |t^r - x^r| \, dt &= \int_a^x (x^r - t^r) \, dt + \int_x^b (t^r - x^r) \, dt \\
&= \left[\frac{2r x^{r+1}}{r+1} - (a+b) x^r + \frac{\left(a^{r+1}+b^{r+1}\right)}{r+1} \right]
\end{aligned}$$

and similarly,

$$\begin{aligned}
\int_c^d |s^r - y^r| \, ds &= \int_c^y (y^r - s^r) \, ds + \int_y^d (s^r - y^r) \, ds \\
&= \left[\frac{2r y^{r+1}}{r+1} - (c+d) y^r + \frac{\left(c^{r+1}+d^{r+1}\right)}{r+1} \right].
\end{aligned}$$

If we take $r \in (-\infty, 0) \setminus \{-1\}$, then we obtain

$$\int_a^b |t^r - x^r| \, dt = -\left[\frac{2r x^{r+1}}{r+1} - (a+b) x^r + \frac{\left(a^{r+1}+b^{r+1}\right)}{r+1} \right]$$

and similarly,

$$\int_c^d |s^r - y^r| \, ds = -\left[\frac{2r y^{r+1}}{r+1} - (c+d) y^r + \frac{\left(c^{r+1}+d^{r+1}\right)}{r+1} \right].$$

Making use of (1.9), we get (1.7).

By using the first inequality in (1.3) for $r = -1$, it follows that

$$\begin{aligned}
& \left| s^{-1} t^{-1} f(x,y) - y^{-1} t^{-1} f(x,s) - x^{-1} s^{-1} f(t,y) + x^{-1} y^{-1} f(t,s) \right| \\
\leq & \left\| l_1 f_{uv} + l_2 f_u + l_3 f_v + f \right\|_\infty \left| x^{-1} - t^{-1} \right| \left| y^{-1} - s^{-1} \right|
\end{aligned}$$

if $l_1 f_{uv} + l_2 f_u + l_3 f_v + f \in L_\infty$. Integrating this inequality, we get

$$
\begin{aligned}
(1.10)\quad & \left| \left(\ln\frac{b}{a}\right)\left(\ln\frac{d}{c}\right) f(x,y) - y^{-1}\left(\ln\frac{b}{a}\right)\int_c^d f(x,s)ds \right. \\
& \left. -x^{-1}\left(\ln\frac{d}{c}\right)\int_a^b f(t,y)dt + x^{-1}y^{-1}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq & \int_a^b\int_c^d \left| s^{-1}t^{-1}f(x,y) - y^{-1}t^{-1}f(x,s) - x^{-1}s^{-1}f(t,y) + x^{-1}y^{-1}f(t,s) \right| dsdt \\
\leq & \left\| l_1 f_{uv} + l_2 f_u + l_3 f_v + f \right\|_\infty \int_a^b\int_c^d \left| x^{-1} - t^{-1} \right| \left| y^{-1} - s^{-1} \right| dsdt.
\end{aligned}
$$

By simple computation,

$$
\begin{aligned}
(1.11)\quad \int_a^b \left| x^{-1} - t^{-1} \right| dt &= \int_a^b \left| \frac{1}{x} - \frac{1}{t} \right| dt \\
&= \int_a^x \left(\frac{1}{t} - \frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x} - \frac{1}{t}\right) dt \\
&= 2\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right)
\end{aligned}
$$

and similarly

$$
(1.12)\quad \int_c^d \left| y^{-1} - s^{-1} \right| ds = 2\left(\ln\frac{y}{\sqrt{cd}} + \frac{\frac{c+d}{2} - y}{y}\right).
$$

By using (1.11) and (1.12) in (1.10), we get the desired inequality (1.8).

Now, we assume that the inequality (1.8) holds with a constant $C > 0$, i.e.

$$
\begin{aligned}
(1.13)\quad & \left| \left(\ln\frac{b}{a}\right)\left(\ln\frac{d}{c}\right) f(x,y) - y^{-1}\left(\ln\frac{b}{a}\right)\int_c^d f(x,s)ds \right. \\
& \left. -x^{-1}\left(\ln\frac{d}{c}\right)\int_a^b f(t,y)dt + x^{-1}y^{-1}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq & \; C \left\| l_1 f_{uv} + l_2 f_u + l_3 f_v + f \right\|_\infty \\
& \times \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right)\left(\ln\frac{y}{\sqrt{cd}} + \frac{\frac{c+d}{2} - y}{y}\right)
\end{aligned}
$$

for any $(x, y) \in \Delta$. If we choose in (1.13), $f(x,s) = f(t,y) = f(t,y) = f(t,s) = 1$, $(t,s) \in \Delta$, the it follows that

$$\begin{aligned} &\left| \left(\ln \frac{b}{a}\right)\left(\ln \frac{d}{c}\right) - \left(\ln \frac{b}{a}\right)\frac{(d-c)}{y} \right. \\ &\left. - \left(\ln \frac{d}{c}\right)\frac{(b-a)}{x} + \frac{(b-a)(d-c)}{xy} \right| \\ \leq\ & C\left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right)\left(\ln \frac{y}{\sqrt{cd}} + \frac{\frac{c+d}{2} - y}{y}\right) \end{aligned} \tag{1.14}$$

for any $(x, y) \in \Delta$. Making $x = a$ and $y = c$ in (1.14) produces the inequality

$$\begin{aligned} &\left| \left(\ln \frac{b}{a}\right)\left(\ln \frac{d}{c}\right) - \left(\ln \frac{b}{a}\right)\frac{(d-c)}{c} \right. \\ &\left. - \left(\ln \frac{d}{c}\right)\frac{(b-a)}{a} + \frac{(b-a)(d-c)}{ac} \right| \\ \leq\ & C\left(\frac{1}{4}\left(\ln \frac{b}{a}\right)\left(\ln \frac{d}{c}\right) - \left(\ln \frac{b}{a}\right)\frac{(d-c)}{4c} - \left(\ln \frac{d}{c}\right)\frac{(b-a)}{4a} + \frac{(b-a)(d-c)}{4ac}\right) \end{aligned}$$

which implies that $C \geq 4$. This proves the sharpness of the constant 4 in (1.8). □

example 1.5. *Let us consider function f defined as $f(u,v) = u^2v^2$. Then we have*

$$f_u(u,v) = 2uv^2,\ \ f_v(u,v) = 2vu^2 \text{ and } f_{uv}(u,v) = 4uv.$$

If we take $a = 1$, $b = 3$, $c = 2$, $d = 4$ and $r = 1$, we get

$$\begin{aligned} &\left| 24x^2y^2 - \frac{224}{3}x^2y - 52xy^2 + \frac{1456}{9}xy \right| \\ \leq\ & \left\| u^2v^2 \right\|_\infty \left(x^2 - 4x + 5\right)\left(y^2 - 6y + 10\right) \end{aligned} \tag{1.15}$$

In particular, if we take $r = -1$, we deduce

$$\begin{aligned} &\left| (\ln 2)^2 x^2y^2 - \frac{56 \ln 2}{3}x^2y^{-1} - \frac{26 \ln 2}{3}x^{-1}y^2 + \frac{1456}{9}x^{-1}y^{-1} \right| \\ \leq\ & 36 \left\| u^2v^2 \right\|_\infty \left(\ln \frac{x}{\sqrt{3}} + \frac{2-x}{x}\right)\left(\ln \frac{y}{\sqrt{8}} + \frac{3-y}{y}\right). \end{aligned} \tag{1.16}$$

If we chose $x = 2$ and $y = 3$ in (1.15) and (1.16), we obtain

$$2.67 \leq 144$$

and

$$39.27 \leq 39.42.$$

Theorem 1.6. *$f : \Delta \to \mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s) \in \Delta$ with $0 < a < b,\ 0 < c < d$. If $r \in \mathbb{R}$, $r \neq 0$ and $l_1 f_{uv} - rl_2 f_u - rl_3 f_v + r^2 f \in L_\infty(\Delta)$, then for any $(t,s), (x,y) \in \Delta$ with*

$x \neq y \neq t \neq s$, we have

$$\begin{aligned}
(1.17)\quad & \left| \frac{f(x,y)}{x^r y^r}(b-a)(d-c) - \frac{(b-a)}{x^r}\int_c^d \frac{f(x,s)}{s^r}ds \right. \\
& \left. - \frac{(d-c)}{y^r}\int_a^b \frac{f(t,y)}{t^r}dt + \int_a^b\int_c^d \frac{f(t,s)}{s^r t^r}dsdt \right| \\
\leq\; & \frac{1}{r^2}\left\| l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \right\|_\infty \left[\frac{2x^{1-r} - a^{1-r} - b^{1-r}}{1-r} - \frac{(a+b-2x)}{x^r} \right] \\
& \times \left[\frac{2y^{1-r} - c^{1-r} - d^{1-r}}{1-r} - \frac{(c+d-2y)}{y^r} \right]
\end{aligned}$$

for $r \neq 1$. Also, for $r = 1$, we have

$$\begin{aligned}
(1.18)\quad & \left| \frac{f(x,y)}{xy}(b-a)(d-c) - \frac{(b-a)}{x}\int_c^d \frac{f(x,s)}{s}ds \right. \\
& \left. - \frac{(d-c)}{y}\int_a^b \frac{f(t,y)}{t}dt + \int_a^b\int_c^d \frac{f(t,s)}{st}dsdt \right| \\
\leq\; & 4\left\| l_1 f_{uv} - l_2 f_u - l_3 f_v + f \right\|_\infty \\
& \times \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x} \right)\left(\ln\frac{y}{\sqrt{cd}} + \frac{\frac{c+d}{2} - y}{y} \right)
\end{aligned}$$

for $(x,y) \in \Delta$, provided $l_1 f_{uv} - l_2 f_u - l_3 f_v + f \in L_\infty(\Delta)$. The constant 4 is best possible in (1.18).

Proof. By using the inequality (1.4), we get

$$\begin{aligned}
& \left| \frac{f(x,y)}{x^r y^r} - \frac{f(x,s)}{x^r s^r} - \frac{f(t,y)}{y^r t^r} + \frac{f(t,s)}{s^r t^r} \right| \\
\leq\; & \left| \int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}} \left| uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v) \right| dvdu \right| \\
\leq\; & \sup_{(u,v)\in[t,x]\times[s,y]} \left| uv f_{uv}(u,v) - ru f_u(u,v) - rv f_v(u,v) + r^2 f(u,v) \right| \left| \int_t^x \int_s^y \frac{1}{u^{r+1}v^{r+1}} dvdu \right| \\
=\; & \frac{1}{r^2}\left\| l_1 f_{uv} - r l_2 f_u - r l_3 f_v + r^2 f \right\|_\infty \left| \frac{1}{x^r} - \frac{1}{t^r} \right| \left| \frac{1}{y^r} - \frac{1}{s^r} \right|
\end{aligned}$$

for $(x,y),(t,s)\in\Delta$. Integrating this inequality with respect to $(t,s)\in\Delta$, we get

$$\begin{aligned}
(1.19)\qquad & \left|\frac{f(x,y)}{x^r y^r}(b-a)(d-c)-\frac{(b-a)}{x^r}\int_c^d \frac{f(x,s)}{s^r}ds\right.\\
& \left.-\frac{(d-c)}{y^r}\int_a^b \frac{f(t,y)}{t^r}dt+\int_a^b\int_c^d \frac{f(t,s)}{s^r t^r}dsdt\right|\\
\leq\ & \int_a^b\int_c^d \left|\frac{f(x,y)}{x^r y^r}-\frac{f(x,s)}{x^r s^r}-\frac{f(t,y)}{y^r t^r}+\frac{f(t,s)}{s^r t^r}\right|dsdt\\
\leq\ & \frac{1}{r^2}\left\|l_1 f_{uv}-rl_2 f_u-rl_3 f_v+r^2 f\right\|_\infty \int_a^b\int_c^d \left|\frac{1}{x^r}-\frac{1}{t^r}\right|\left|\frac{1}{y^r}-\frac{1}{s^r}\right|dsdt
\end{aligned}$$

for $r\in\mathbb{R}$, $r\neq 0$. By simple computation, if we take $r\in(0,\infty)\backslash\{1\}$, then we obtain

$$\begin{aligned}
(1.20)\qquad & \int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt\\
=\ & \int_a^x \left(\frac{1}{t^r}-\frac{1}{x^r}\right)dt+\int_x^b \left(\frac{1}{x^r}-\frac{1}{t^r}\right)dt\\
=\ & \left[\frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r}-\frac{(a+b-2x)}{x^r}\right]
\end{aligned}$$

and similarly

$$(1.21)\qquad \int_c^d \left|\frac{1}{y^r}-\frac{1}{s^r}\right|ds=\left[\frac{2y^{1-r}-c^{1-r}-d^{1-r}}{1-r}-\frac{(c+d-2y)}{y^r}\right].$$

for any $(x,y)\in\Delta$. On the other hand, for $r>0$, we have

$$(1.22)\qquad \int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt=-\left[\frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r}-\frac{(a+b-2x)}{x^r}\right]$$

and

$$(1.23)\qquad \int_c^d \left|\frac{1}{y^r}-\frac{1}{s^r}\right|ds=-\left[\frac{2y^{1-r}-c^{1-r}-d^{1-r}}{1-r}-\frac{(c+d-2y)}{y^r}\right]$$

for any $(x,y)\in\Delta$. By using (1.20)-(1.23) in (1.19), we get the desired inequality (1.17).

For $r=1$, proceeding as in the proof of Theorem 1.4, we also obtain the inequality (1.18). $\square$

Conclusion

In this paper, we have investigated several Pompeiu's type inequlity for complex valued absolutely continuous functions with double integrals involving functions of two independent variables. Firstly, we obtained an important identity as Lemma 1.1 and by using this lemma we established the new versions of the inequalities involving several differentiable functions.

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Department of Mathematics, Faculty of Science and Arts, Düzce University. Konuralp Campus, Düzce-Turkey
E-mail address: sarikayamz@gmail.com

Department of Mathematics, Faculty of Science, Bartin University. Bartin-Turkey
E-mail address: erdem1627@gmail.com

Department of Mathematics, Faculty of Science and Arts, Düzce University. Konuralp Campus, Düzce-Turkey
E-mail address: yaldizhatice@gmail.com