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ON THE SIMSON–WALLACE THEOREM

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Abstract. The paper deals with the well-known Simson–Wallace theorem and its Gergonne's generalization. In this article we proceed with the generalization of the Simson–Wallace theorem a bit further. We inspect the validity of previous theorems when affine transformation is applied. This leads to deriving a few interesting insights. Throughout the paper we used both synthetic and analytical methods.

1. Introduction

This section serves as an introduction into the topic of the Simson–Wallace theorem. If the reader already possesses the knowledge of this topic, he might skip it to the next section.

The Simson–Wallace theorem describes an interesting property regarding the points on a circumcircle of a triangle [4], [5].

Theorem 1.1 (Simson–Wallace). *Let ABC be a triangle and P a point in a plane. The feet of perpendicular lines onto the sides of the triangle are collinear iff P lies on the circumcircle of ABC, Fig.1.*

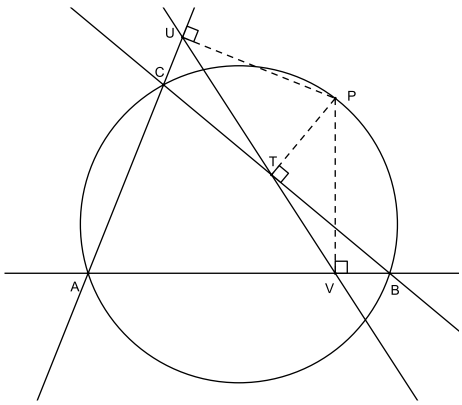

Figure 1. Simson–Wallace theorem — points T, U, V are collinear

The Simson–Wallace theorem has several generalizations. J. D. Gergonne generalized the Simson–Wallace theorem as follows:

Theorem 1.2 (Gergonne). *Let ABC be a triangle and P a point in a plane. The feet of perpendicular lines onto the sides of the triangle form a triangle with*

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a constant area iff P lies on the circle which is concentric with the circumcircle of ABC, Fig. 2.

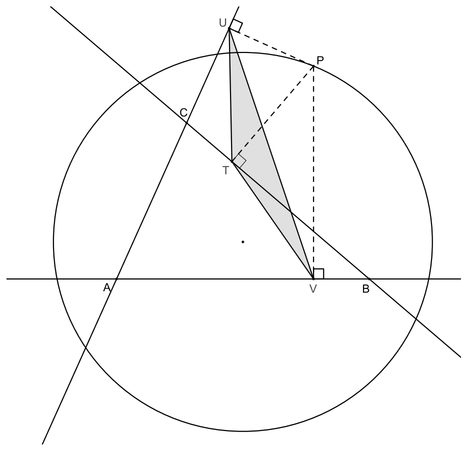

FIGURE 2. Gergonne's generalization — triangle TUV has a constant area

Gergonne's generalization formulates the necessary and sufficient condition for the pedal triangle of P with respect to $\triangle ABC$ to have a constant area. If the pedal triangle degenerates into the line, we get the Simson–Wallace theorem.

In this article we proceed with the generalization of the Simson–Wallace theorem a bit further and we inspect the validity of previous theorems when affine transformation is applied. This leads to deriving a few interesting insights.

2. Affine generalization of the Simson–Wallace theorem

First we recall several important properties of affine transformations in a plane which will be used:

Lemma 2.1. *Affine transformation preserves collinearity, i.e., image of a line in affine transformation $\mathcal{A}$ is also a line or a point, or: if three points are collinear, then their images are collinear as well.*

We may generalize the Simson–Wallace theorem with respect to this lemma in this way:

Theorem 2.2. *Let ABC be a triangle and P a point in a plane. The images of the feet of the perpendicular lines in an affine transformation $\mathcal{A}$ are collinear iff P lies on the circumcircle of ABC.*

Let us leave Theorem 2.2 for a moment and show its consequences later. Another important property of an affine transformation is as follows:

Lemma 2.3. *Let $\mathcal{A}$ be an affine transformation. If oriented areas of two triangles in a coordinate system Oxy are in the ratio $S_1 : S_2 = k$ then the ratio of the oriented areas of the images of these triangles is preserved.*

In a special case, if these two oriented areas are equal, then the oriented areas of their images are equal as well.

Applying this lemma, we are even able to generalize the Gergonne's theorem in a similar way:

Theorem 2.4. *Let ABC be a triangle and P a point on a circle concentric with the circumcircle of a triangle ABC. Let TUV be a pedal triangle of the point P. Then the area of the triangle $T'U'V'$, whose vertices are images of pedals in affine transformation $\mathcal{A}$, is for all points on the circle constant.*

This theorem is clearly a generalization of the previous Theorem 2.2. Substituting

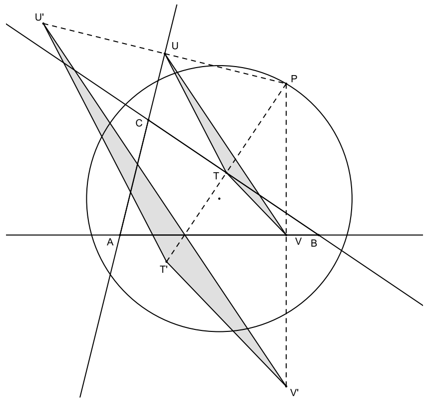

FIGURE 3. Affine generalization of the theorem of Gergonne – reflection

affine transformation in the Theorem 2.4 gradually by its specific examples, we show a geometrical application and obtain:

Consequence 1. We assume the affine transformation to be a homothety with center in P and a scale ratio of $\kappa = 2$. As we easily see, this case is identical with the reflection of this point with respect to the sides of a triangle ABC, Fig. 3. Thus the points T', U', V' are images of P in this reflection. For the area of the so formed triangle $T'U'V'$ we get $S' = 4S$.

As special case we assume the case mentioned in the Theorem 2.2, when these

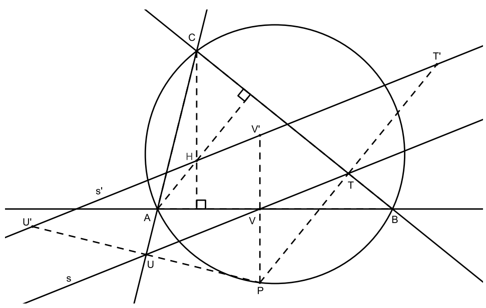

FIGURE 4. Affine generalization of the Simson–Wallace theorem – reflection

three points are collinear. We obtain:

Theorem 2.5. *Let ABC be a triangle and P a point in a plane. If P lies on its circumcircle, then its images in the reflection with respect to the sides of the triangle ABC are collinear being on the line s' which is parallel to the Simson line s, Fig. 4.*

Let us explore the properties of the line s'. When we choose an arbitrary position of P on the circumcircle of the triangle ABC, then we see that the line always passes

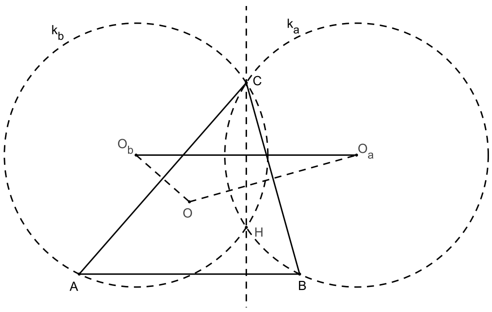

FIGURE 5. Circles k_a and k_b pass through the orthocenter H

through the orthocenter of the triangle. We state a theorem:

Theorem 2.6. *Let ABC be a triangle and P a point in a plane. If P lies on its circumcirle, then its images in reflection with respect the sides of the triangle ABC and the orthocenter of the triangle are collinear, Fig. 4.*

Proof. We will give a synthetic proof of the theorem. First we show that three circles k_a, k_b and k_c which are symmetric with the circumcircle k of a triangle ABC pass through the orthocenter H of ABC, Fig. 5.
Let O be the circumcenter of ABC, H the intersection of the circles k_a and k_b and let O_a, O_b be centers of k_a, k_b. As O_a and O_b are symmetric with O with respect

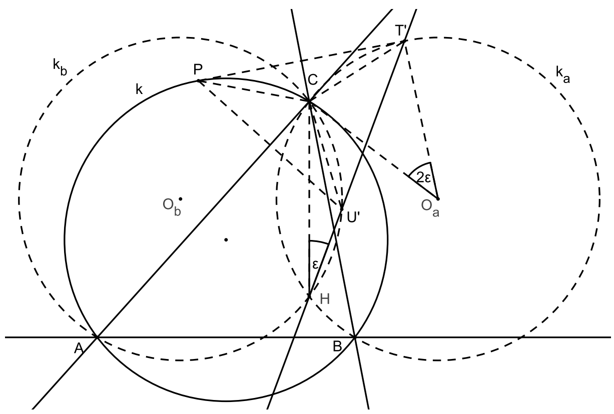

FIGURE 6. Points H, U' and T' are collinear

to the sides BC and AC then $O_aO_b \parallel AB$ and $|O_aO_b| = |AB|$. It implies that CH must be the perpendicular bisector of the segment O_aO_b. Thus $CH \perp AB$.
Now we will prove, that points H, U' and T' are collinear, Fig. 6, from which our statement follows. We show that the angles between lines CH, HT' and CH, HU' are the same. If we denote $\angle CHT' = \epsilon$ then by the inscribed angle theorem $\angle CS_aT' = 2\epsilon$. Similarly, $\angle CHU' = 1/2\angle CS_bU' = \epsilon$ as $|CU'| = |CT'|$. The theorem is proved. □

Consequence 2. A second interesting consequence goes by when we choose affine transformation in the Theorem 2.4 to be a transformation composed of a rotation and a homothety. Rotating T, U, V around P about the angle θ and mapping in a homothety centered in P with a coefficient $k = \frac{1}{\cos\theta}$, we obtain points T', U', V' lying on the sides of a triangle and the angle α between the lines $T'P$, $U'P$, $V'P$ and the sides of a triangle ABC is constant. Then the next theorem holds:

Theorem 2.7. *Given a triangle ABC, a point P, an oriented angle α, $\alpha \in (0, \pi)$ and points T', U' and V' on the sides BC, AC and AB of the triangle ABC respectively. If P lies on the circumcircle of ABC and $\angle(T'P, BC) = \angle(U'P, AC) = \angle(V'P, AC) = \alpha$, then the points T', U' and V' are collinear, Fig. 7.*

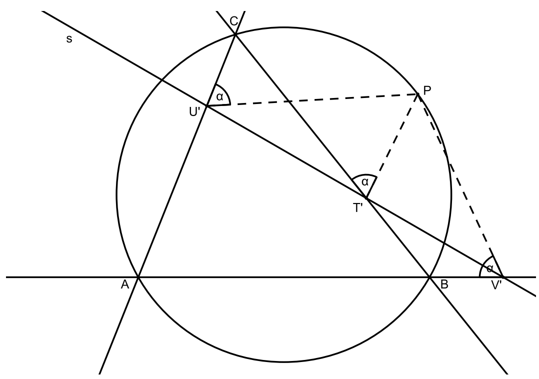

FIGURE 7. Affine generalization – arbitrary angle

Similarly, assuming P to lie on a circle concentric with the circumcircle of a triangle ABC then the oriented area of a triangle $T'U'V'$ is constant for all the points of this circle.[1]

3. Nine point circle

At this point we remark that not only the line s' which is parallel to the Simson line s, but even the original Simson line is related to the orthocenter of the triangle. To explore it, we introduce a coordinate system and use analytical method.

Adopt a rectangular coordinate system, where $A = [0, 0]$, $B = [b, 0]$, $C = [c_1, c_2]$ and $P = [p, q]$, Fig. 8. To the feet T, U, V of perpendiculars from P to the sides of

[1]For the area of the so formed triangle it might be easily proven that $S' = \frac{1}{\sin^2\alpha} S$.

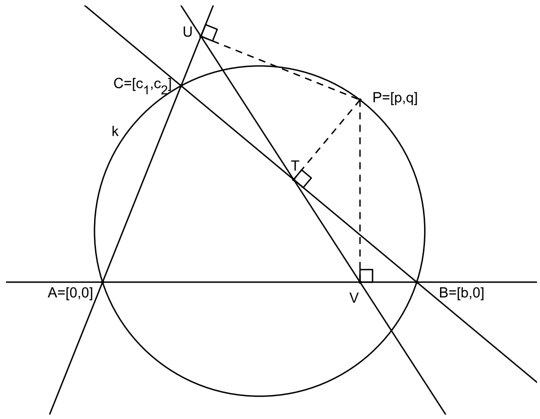

FIGURE 8. Introduction of a coordinate system

ABC we assign the coordinates $T = [t_1, t_2]$, $U = [u_1, u_2]$ and $V = [v_1, v_2]$.
We see that $v_1 = p$ and $v_2 = 0$. Further we easily find that:

$$u_1 = \frac{c_1(pc_1 + qc_2)}{c_1^2 + c_2^2}, \quad u_2 = \frac{c_2(pc_1 + qc_2)}{c_1^2 + c_2^2},$$

and

$$t_1 = \frac{p(c_1 - b)^2 + bc_2^2 + qc_2(c_1 - b)}{(c_1 - b)^2 + c_2^2}, \quad t_2 = \frac{pc_2(c_1 - b) + qc_2^2 - bc_2(c_1 - b)}{(c_1 - b)^2 + c_2^2},$$

where for the point $P = [p, q]$

$$c_2p^2 + c_2q^2 - bc_2p + bc_1q - qc_1^2 - qc_2^2 = 0 \tag{3.1}$$

holds.
It is obvious that (3.1) is the equation of the circumcircle k of ABC. We will also need the equation of the Simson line s which is as follows

$$x(pc_1 + qc_2) + y(pc_2 - qc_1) - p(pc_1 + qc_2) = 0, \tag{3.2}$$

where variables p, q obey (3.1).
Finally for the coordinates of the orthocenter H we get $H = \left[c_1, \frac{c_1(b-c_1)}{c_2}\right]$.

Now assuming the midpoint S of a segment HP with endpoints in the orthocenter H and in a point P on the circumcircle of ABC, Fig. 9, with coordinates $S = \left[\frac{p+c_1}{2}, \frac{qc_2+c_1(b-c_1)}{2c_2}\right]$, we see that it satisfies the equation (3.2) of the Simson line, thus it lies on it.[2] Really, if we set in the coordinates of the point S into (3.2), we get

$$c_1(c_2p^2 + c_2q^2 - bc_2p + bc_1q - qc_1^2 - qc_2^2),$$

which equals, with respect to (3.1), zero.

Simultaneously, eliminating the parameters p and q from the coordinates of the

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[2]This might also be proven as follows: image of a line s' in a homothety centered at P and with a coefficient $\kappa = 0.5$ is the original Simson line. The midpoint S, which is the image of the orthocenter V, thus lies on the Simson line.

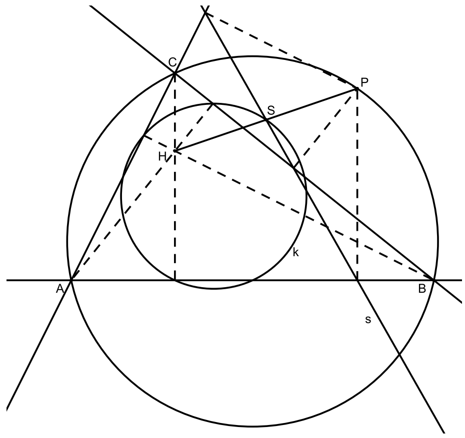

FIGURE 9. The intersection S of the Simson line s and the nine-point circle k bisects the segment PH

point S and substituting it into the condition that P lies on the circumcircle of ABC, we obtain the equation of the locus of these midpoints

$$2\,c_2x^2 - 2\,c_2c_1x + 2\,c_2y^2 - c_1yb + {c_1}^2y - c_2xb + c_2c_1b - {c_2}^2y = 0. \tag{3.3}$$

We see that this equation describes a circle. Altering this equation, a center-radius form of the circle might be obtained

$$\left(x - \frac{2c_1 + b}{4}\right)^2 + \left(y - \frac{4{c_2}^2 + bc_1 - {c_1}^2}{c_2}\right)^2 = \frac{\left({c_1}^2 + {c_2}^2\right)\left((c_1 - b)^2 + {c_2}^2\right)}{16\,{c_2}^2}. \tag{3.4}$$

Let us focus on, in what relation is this circle (3.4) with the original triangle ABC. Substituting into its equation, we see that several distinguished points lie on the circle — e.g. feet of altitudes or midpoints of sides satisfy the equation (3.4). Thus this equation describes the *nine-point circle* which is often called the *Feuerbach circle*, [1], [5]. This midpoint S lies not only on the Simson line, but also on the Feuerbach circle, Fig. 9.

If we become aware of the full extent of the previous statement, we can define, apart from the usual definition, the nine-point circle as follows:

Theorem 3.1. *The nine-point circle is a locus of midpoints of segments connecting the orthocenter with the points on a circumcircle of the triangle.*

Remark 3.2. We can see, that the Theorem 2.6 is a special case of the less known Hagge theorem. K. Hagge discovered in 1907 a construction of a circle always passing through the orthocenter of a given triangle [3].

Theorem 3.3 (Hagge). *Let ABC be a triangle and P a point in a plane different from the vertices of the triangle ABC. Let us denote by K the intersection of the line AP with the circumcircle k and next denote its image in the reflection with respect to the line BC as K'. Similarly we obtain points L' and M'. Then the orthocenter H lies on a circle passing through the points K', L' and M', Fig. 10.*

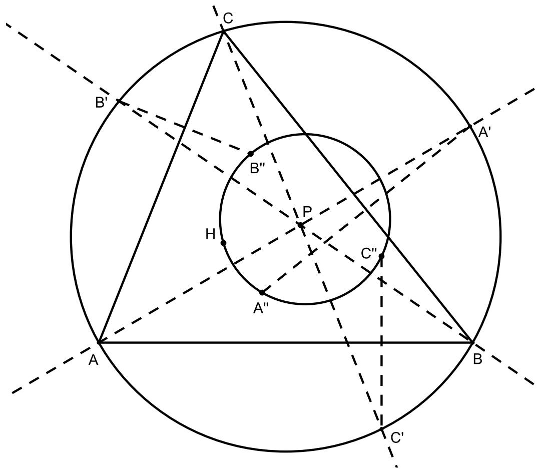

FIGURE 10. Hagge theorem

If we assume in this theorem that the point P lies on the circumcircle of ABC, then its respective Hagge circle degenerates into a parallel line to the Simson line of P (infinite radius circle). Further, if P is the orthocenter H itself, then the circle degenerates into a point (zero-radius circle). Consequently, this theorem might be considered in the certain sense as a generalization of the Simson–Wallace theorem.

4. Conclusion

A few properties of the well known Simson–Wallace theorem were discussed. The novelty of the paper consists in the description of these properties by an affine transformation. Moreover analytical approach enables to define the nine-point circle of a triangle in the less known way by the midpoints of segments connecting an arbitrary point of the circumcircle and the orthocenter of the triangle.

References

[1] Baker, H.F.: An introduction to plane geometry: with many examples. Cambridge University Press, Cambridge, 1943.
[2] Budinský, B.: Analytická a diferenciální geometrie. SNTL, Praha, 1983.
[3] Bradley, J.Ch., Smith, C.G.: On a Construction of Hagge. Forum Geometricorum, 7 (2007), 231–247. [online]: http://forumgeom.fau.edu/FG2007volume7/FG200730index.html
[4] Coxeter, H.S.M.: Introduction to geometry. John Wiley and Sons, New York, 1989.
[5] Coxeter, H.S.M., Greitzer, S.L.: Geometry revisited (New Mathematical Library). The Mathematical Association of America, Washington, 1967.
[6] Pech, P.: Klasické vs. počítačové metody při řešení úloh v geometrii. Jihočeská univerzita, České Budějovice, 2005.
[7] Rektorys, K. et al.: Přehled užité matematiky. SNTL, Praha, 1988.
[8] Švrček, J.: Vybrané kapitoly z geometrie trojúhelníka. Karolinum, Praha, 2004.

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