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CONSTRUCTIONS OF QUADRILATERAL MESHES: A COMPARATIVE STUDY

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ABSTRAKT. Polygonal meshes represent important geometric structures with a large number of applications. The study of polygonal meshes is motivated by many processing tasks in automotive/aerospace industry, egineering, architecture, engineering, construction industry, and industrial design. Much of literature on polygonal representations focuses on quadrilateral meshes which are composed of quadrilaterals as they possess several advantages compared to triangle meshes. In this short contribution we present a comparative study of known methods for constuctions of quadrangulations of various classes and for different purposes. We suggest a new method for computing all unique quadrilateral meshes of a certain class based on sequential construction.

Introduction

Polygonal meshes are widely used as a stepping stone for representations of 2D and 3D objects in many computational processes. The practical applications of polygonal meshes include computer-aided architectural and industrial design, [8], engineering and construction industries, [14], reverse engineering, [7], the digitization of real objects using 3D scanning, [1], digital surface reconstruction from point clouds, [16], the replication of the shapes of real-world objects using 3D printing, [10], computer graphics, [9] and many more. The creation of polygonal meshes is based on the idea of cell decomposition where triangles and quadrilaterals are the most common shapes of cells in this decomposition. Quadrilateral meshes, which are composed of quadrilaterals, have been widely used in computer graphics because some geometric tasks are better solved using quadrilateral meshes than triangle meshes. Quadrilateral meshes have significant advantages comparing to triangle meshes in many cases. There are some requirements which make geometry processing on quadrilateral meshes much more difficult than on triangle meshes. For example we can require mesh cells to be rectangular or even nearly squared. Quadrilaterals are more delicate and less adaptive structures than triangles that is the study of quadrilateral meshes represents challenges to the both researchers and practitioners.

This contribution is specifically aimed at the comparison of several known methods for constructions of quadrilateral meshes of certain classes which are based on different bases. In addition to these algorithms for quadrangulations we suggest a new method of incremental construction of quadrilateral mesh. Our aim is to enumerate topologically unique meshes of certain class. More precisely, we propose

Key words and phrases. Quadrilateral meshes, quadrilateral, n-sided planar region, incremental construction.

an algorithm that enumerates all possible quad layouts with respect to the number of internal vertices with the given valences.

1. Terminology

Let us introduce the underlying definitions and basic terminology. A quadrilateral mesh, [5, 11], is a triple (V, E, Q) where V is a set of vertices, E is a set of edges, and Q is a set of quadrilaterals. There exists an embedding of (V, E, Q) into 2D plane such that each vertex is represented as a point in the plane and each edge is represented as a curve in the plane, so that curves connect vertices and each quad is depicted in the plane as a quadrilateral. In our study we furthermore assume only quadrilateral meshes that form a connected, conforming (i.e. free from T-junctions), orientable *2D manifold with boundary*, [5], i.e. we define quadrilateral meshes for segmentation of simply connected planar domains, see Figure 1. The mesh topology is discussed in more detail in [6].

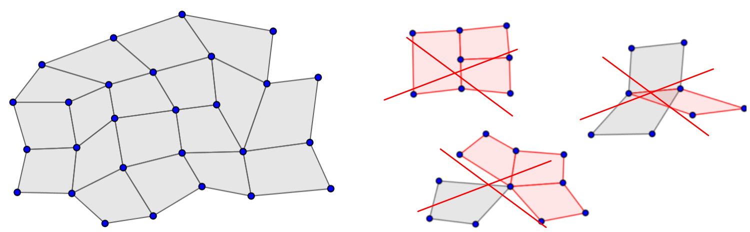

Figure 1. An example of a quadrilateral mesh - a 2D manifold with boundary (on the left), three examples of non-manifold meshes - T-junction, an edge shared by three quadrilaterals, and a vertex shared by two otherwise unconnected sets of quadrilaterals (on the right).

Regarding the terminology, an edge of the mesh with two incident quads is said to be *internal*, while an edge with just one incident quad is said to be *boundary*. A vertex of an internal edge is also said to be *internal*, otherwise it is said to be *boundary*. The *valence* of a vertex is the number of edges incident to that vertex.

An internal vertex in a quadrilateral mesh is *regular* if it has valence 4. For quadrilateral meshes, we distinguish three types of regular boundary vertices. A regular *non-corner* boundary vertex in a quadrilateral mesh has valence 3, convex *corner* boundary vertex has valence 2, and concave *corner* boundary vertex has valence 4. A vertex that is not regular is denoted as an *irregular* vertex.

2. State of the Art in Quadrilateral Meshes

Regarding the state of the art in quadrilateral meshes, there exist several methods and algorithms for constructing quadrangulations for an n-sided planar region which use different approaches. Typically, the approaches consist of two phases - the topological phase where the connectivity of the mesh is determined and the geometrical phase where the positions of the vertices in space are solved.

Among the oldest approaches the paving techniques belong which are based on iteratively paving rows of elements to the interior of a region's boundary, [2]. Using this method we can quadrangulate arbitrary n-sided planar region. The generated mesh has nearly squared elements and mesh boundary tend to follow geometric contours of the given boundary. Simple example is given in Figure 2, the rows eventually fill the region from the boundary inward.

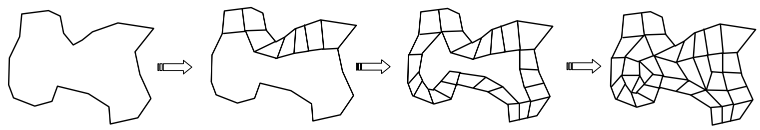

FIGURE 2. An example of a simple paving sequence.

Some methods use dual graph, [15], of quad mesh for the representation and enumeration of a set of all possible quad meshes, [12]. Dual of quadrilateral mesh is slightly different from the standard because we do not consider unbounded face in quadrilateral mesh, see Figure 3. Dual of quadrilateral mesh provides a simple method for defining the basic connectivity rules for all quadrilateral meshes because of its unique properties. Then the meshing algorithm is based on enumerating all graphs of a certain class.

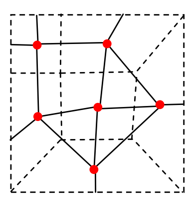

FIGURE 3. Dual of a quadrilateral mesh.

Another methods adaptively fills an n-sided region with a triangle mesh and then merges triangles to quads, [4]. Generating a triangle mesh is a well-studied problem and many meshing strategies have been proposed and studied. The method for constructing of quadrilateral mesh starts with a given triangular mesh of n-sided region and aims at combining triangles into quadrilaterals, see Figure 4. We can use two related merging processes. The triangle merging procedure is controlled by the quadrilateral quality and may lead to mixed triangular-quadrilateral meshes, or it starts from the boundary and moves to the interior, and results in pure quadrilateral meshes.

It is also possible to quadrangulate an n-sided region with prescribed numbers of edge subdivisions at the boundary, see Figure 5. This method is based on the idea to provide a complete set of topological patterns that covers all possible inputs, [17]. The problem of determining the correct mesh for a given n-sided planar region can be formulated as a small integer linear problem. Then it is possible to explore the sets of possible quadrangulations for the given input.

Some another approaches attempt to find a topology with the fewest irregular vertices, [13], see Figure 6.

Another methods and algorithms and the advantages and problems of some techniques for quadrilateral meshes are discussed in details in the survey [3].

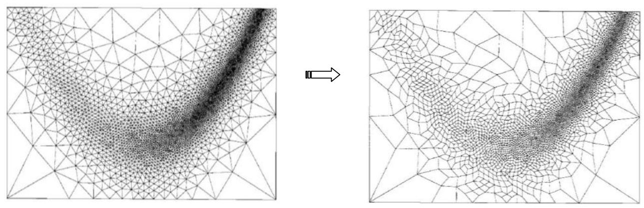

FIGURE 4. A quadrilateral mesh generated using triangle merging.

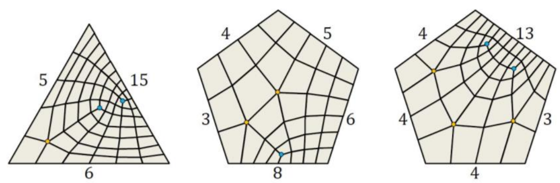

FIGURE 5. Quadrangulations of n-sided regions with prescribed numbers of edge subdivisions at the boundary.

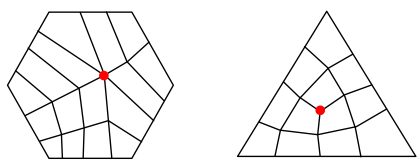

FIGURE 6. Filling n-sided region with the fewest irregular vertices.

3. INCREMENTAL CONSTRUCTION

3.1. **Quadrilateral Meshes of a Certain Class.** For enumerating all possible quadrangulations with respect to some input we have to define some restriction on the types of meshes, otherwise the number of all possibilities is too high. Thus, in our work we consider only valences ≤ 5 for the both boundary and internal vertices. The quadrilateral meshes furthermore satisfy the following invariant:

- At least one vertex of each internal edge is internal.

We assume that the number of internal vertices is specified for exploration all possible unique quadrangulations. We also distinguish the types of valences of these internal vertices, for internal vertices only valences 3, 4, and 5 are allowed. The set of quadrilateral meshes with n_3 internal vertices of valence 3, n_4 internal vertices of valence 4, and n_5 internal vertices of valence 5 is denoted by $M(n_3, n_4, n_5)$.

3.2. **Incremental Construction of Quadrilateral Mesh.** Our goal is to develop an incremental construction of a quadrilateral mesh and enumerate all unique meshes of a certain class. More precisely, an algorithm will enumerate all possible quad layouts with respect to the number of internal vertices with the given valences and the duplications will be filtered out within a further post-processing for computed set of meshes.

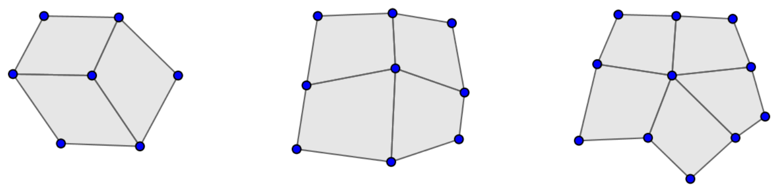

FIGURE 7. Three types of quadrilateral meshes with one internal vertex of valences 3, 4, and 5.

We construct a concrete quadrilateral mesh $M \in M_I(n_3, n_4, n_5)$, i.e. a mesh with n internal vertices where the types of valences are distinguished, incrementally starting from a trivial mesh with one internal vertex, see Figure 7. In each step of construction the number of internal vertices of a quadrilateral mesh M is increased by one by adding new elements into a quadrilateral mesh or by additional modifying of a quadrilateral mesh using *mesh operations*. A quadrilateral mesh always satisfies the invariant, i.e. at least one vertex of each internal edge is internal, before and after modification.

4. Conclusion

We discussed several methods for constructions of quadrilateral meshes based on different bases. We presented a framework of a new constructing method for generating of all possible unique quadrilateral meshes of a certain class for the given number of internal vertices. We will focus on the formulation of mesh operations and on implementation of an algorithm. The experimental evaluation will be provided and targeted on the number of meshes in different sets $M_I(n_3, n_4, n_5)$.

Regarding the future work we want to extend our method for hexahedral meshes and suggest similar incremental construction in the 3D space.

References

[1] 3DSYSTEMS: *3D Scanners - A Guide to 3D Scanner Technology,* http://www.rapidform.com, 1989, [retrieved July 2015]

[2] Blacker, T., Stephenson, M.: *Paving: A New Approach to Automated Quadrilateral Mesh Generation,* Numerical Methods in Engineering, 32(4), 811–847, 1991

[3] Bommes, D., Lévy, B., Pietroni, N., Puppo, E., Silva, C., Tarini, M. and Zorin, D.: *Quad-Mesh Generation and Processing: a Survey,* Computer Graphics Forum, 32(6), 2013

[4] Borouchaki, H., Frey, P. J.: *Adaptive Triangular-Quadrilateral Mesh Generation,* Numerical Methods in Engineering, 41(5), 915–934, 1998

[5] Botsch, M., Kobbelt, L., Pauly, M., Alliez, P., Lévy, B.: *Polygon Mesh Processing,* A K Peters, Natick, 2010

[6] H. Edelsbrunner: *Geometry and Topology for Mesh Generation,* Cambridge University Press, New York, 2001

[7] Eilam, E.: *Reversing: Secrets of Reverse Engineering,* Wiley Publishing, Indianapolis, 2005

[8] Farin, F., Hoschek, J., Kim, M.-S.: *Handbook of Computer Aided Geometric Design,* Elsevier Science, Amsterdam, 2002
[9] J. D. Foley, A. Dam, S. K. Feiner, J. F. Hughes: *Computer Graphics: Principle and Practice,* Addison-Wesley Publishing Company, 1995
[10] Lipson, H., Kurman, M.: *Fabricated: The New World of 3D Printing,* John Wiley and Sons, Indianapolis, 2013
[11] Mitchell, S. A.: *A Characterization of the Quadrilateral Meshes of a Surface Which Admit a Compatible Hexahedral Mesh of the Enclosed Volume,* 13th Annual Symposium on Theoretical Aspects of Computer Science, 1996
[12] Murdoch, P., Benzley, S., Blacker, T., Mitchell, S. A.: *The Spatial Twist Continuum: A Connectivity Based Method for Representing All-Hexahedral Finite Element Meshes,* Finite Elements in Analysis and Design, 28(2), 137-149, 1997
[13] Nasri, A., Sabin, M., Yasseen, Z.: *Filling N-Sided Regions by Quad Meshes for Subdivision Surfaces,* Computer Graphics Forum, 28(6), 1644-1658, 2009
[14] Pottmann, H., Asperl, A., Hofer, M., Kilian, A.: *Architectural Geometry,* Bentley Institute Press, Exton, 2007
[15] Preparata, F. P., Shamos, M. I.: *Computational Geometry: An Introduction,* Springer-Verlag, New York, 1985
[16] Surynková, P.: *Surface Reconstruction,* Proceedings of the 18th Annual Conference of Doctoral Students, MATFYZPRESS, 2009
[17] Takayama, K., Panozzo, D., Sorkine-Hornung, O.: Pattern-Based Quadrangulation for N-Sided Patches, Computer Graphics Forum, 33(5), 177–184, 2014

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