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NOTES ON PROPERTIES AND APPLICATIONS OF MINKOWSKI POINT SET OPERATIONS

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Dedicated to the memory of prof. RNDr. Václav Medek (23.10.1923 - 31.3.1992), outstanding Slovak mathematician, at 25th anniversary of his death.

Abstract. This paper provides information about some basic properties of Minkowski point set operations - sum, difference and product of two point sets, namely basic geometric objects, i.e. points, lines and curves. Resulting geometric figures are determined analytically by vector maps and their intrinsic geometric properties are derived by means of derivatives of vector functions representing operands of respective Minkowski operations. Interesting examples of generated complex geometric forms and structures are included, with illustrations of their applications.

Introduction

Minkowski sum $\oplus$, Minkowski difference Θ, and Minkowski product $\otimes$ of two point sets are point-wise operations based on sum and product of two points in a basic geometric space. More definitions are available, based on different approach to representation of a point (vector, complex number or quaternion). In this paper we will use the vector based definition of point. Operations are performed on points' position vectors with respect to a fixed reference point (usually, for the sake of convenience and clarity, it is located at the origin of space coordinate system). Minkowski sum of point sets is defined as vector sum and Minkowski product of point sets as exterior (wedge) product of position vectors of all points in the respective sets.

Manifolds represented by their vector maps are considered as infinite sets of points whose position vectors are determined as respective values of the vector functions defining the manifolds. Therefore, operations of Minkowski sum and Minkowski product of two manifolds can be performed as operations on their vector maps. Vector maps of operands in Minkowski point set operations determine form of the vector maps of resulting manifolds. These can be thus easily represented analytically, while their intrinsic geometric properties can be derived by means of differential characteristics of the two operand manifolds. Different properties of background vector operations of vector sum and wedge product of vectors influence intrinsic characteristics of resulting manifolds, too. Position of operand sets with respect to the reference point at the origin of the coordinate system is one of the key characteristics of the classification.

Let us suppose first that none of the operand manifolds determined by the vector map defined on interval in real numbers contains reference point O. Such situations will be considered as special singularities and treated separately.

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1. Minkowski operations on pairs of points

Assuming Cartesian coordinate system, an arbitrary point $p \neq O$ from $\mathbf{E}^n$ is attached a unique n-tuple of Cartesian coordinates $p = (p_1, p_2, ..., p_n)$, while its no-zero position vector be determined as $\mathbf{p} = (p_1, p_2, ..., p_n)$.

Starting from Minkowski sum $\oplus$ and Minkowski product $\otimes$ of two points, the resulting points are determined straightforwardly

$$a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n)$$

$$a \oplus b = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$$

$$a \otimes b = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + ... + a_n\mathbf{e}_n) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + ... + b_n\mathbf{e}_n) =$$

$$= (a_1b_2 - a_2b_1)\,\mathbf{e}_{1\wedge 2} + ... + (a_{n-1}b_n - a_nb_{n-1})\,\mathbf{e}_{n-1\wedge n}$$

while sum of points $a, b \in \mathbf{E}^n$ is again a point in $\mathbf{E}^n$, whereas their product is a point in the space $\mathbf{E}^d$, for $d = n(n-1)/2$.

Bi-vectors

$$\mathbf{e}_{1\wedge 2}, \mathbf{e}_{1\wedge 3}, ..., \mathbf{e}_{1\wedge n}, \mathbf{e}_{2\wedge 3}, \mathbf{e}_{2\wedge 4}, ..., \mathbf{e}_{2\wedge n}, ..., \mathbf{e}_{n-1\wedge n}$$

form ortho-normal basis of the bi-vector space $\wedge^2\left(\mathbf{E}^d\right)$. For dimension $n = 3$, wedge product can be considered as equivalent to the usuall cross vector product. Euclidean space $\mathbf{E}^3$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is isomorphic to the bi-vector space $\wedge^2\left(\mathbf{E}^3\right)\}$ with basis $\{\mathbf{e}_{1\wedge 2}, \mathbf{e}_{1\wedge 3}, \mathbf{e}_{2\wedge 3}\}$, as there exists a regular linear transformation $\phi : \wedge^2\left(\mathbf{E}^3\right) \to \mathbf{E}^3$ mappig one basis to the other one. Matrix of this transformation is the following regular square matrix of rank 3

$$M_\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

For an arbitrary vector $\mathbf{u}$ holds $\phi(\mathbf{u}) = \mathbf{u}.M_\phi$, therefore

$$\phi(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3, \phi(\mathbf{e}_1 \wedge \mathbf{e}_3) = -\mathbf{e}_2, \phi(\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_1.$$

2. Basic definitions

Minkowski sum is believed to be introduced by Herman Minkowski arround 1903, during his close cooperation with David Hilbert in Gottingen. This operation is used in Grassman algebras for various symbolic calculations. Re-discovered by Rida Farouki in 1990-ties it was applied in computer graphics to summing up polygonal regions in plane or convex polyhedra (polytopes) in higher dimensions, calculation of offsets, robot motion planing, etc., see [3] - [5].

Definition 2.1. Minkowski sum of point sets A and B is set S of all such points that are sums of all points $a \in A$ with all points $b \in B$

$$S = A \oplus B = \{a \oplus b, a \in A, b \in B\}.$$

Interesting geometric interpretation of Minkowski sum of two point sets as continuous motion of one set on the boundary of the other without any change of orientation leads to the other equivalnt form of its definition.

Definition 2.2. Minkowski sum of point sets A and B is set S that is the union of all positions of set A translated by all position vectors of points from set B

$$S = A \oplus B = \cup_{b \in B} A^b.$$

Properties of Minkowski sum $\oplus$

1. Commutativity $A \oplus B = B \oplus A$
2. Associativity $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
3. Distributivity $(A \cup B) \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D) \cup (B \oplus C) \cup (B \oplus D)$
4. $\cup P_i \oplus \cup Q_j = \cup(P_i \oplus Q_j, i, j \in N$

Minkowski difference of two point sets can be, reversely, introduced in various not equivalent forms.

Definition 2.3. Minkowski difference of point sets A and B is set D of all such points d whose Minkowski sum with the set B is a subset of set A

$$D = A \Theta B = \{d, d \oplus B \subseteq A\}.$$

Remark 2.4. Formula relating Minkowski sum and Minkowski difference is as follows

$$D = A \Theta B = (A^C \oplus (-B)^C)^C$$

$$A \cup A^C = C, -B \cup (-B)^C = C$$

while A^C and $(-B)^C$ stand for complements of sets A and $-B$ in set C.

Definition 2.5. Minkowski difference of point sets A and B is set D of all such points that are differences of all points $a \in A$ and all points $b \in B$

$$D = A \oplus (-B) = \{a - b, a \in A, b \in B\}.$$

The above definitions determine different sets, therefore in general

$$A \Theta B \neq A \oplus (-B).$$

Operation of Minkowski product was introduced by Rida Farouki in 2001 on complex planar sets (in $C = R \times R$) by means of product of complex numbers representing points in the complex plane. Concept has been extended in 2002 by Weiner and Gu, [8], later in 2003 by Smukler to 4D (in $H = C \times c$) by means of quaternion product, [9]. In 2013 it was generalized to arbitrary dimension n by means of wedge (outer) vector product of points' position vectors, see in [11] - [13], satisfying the following properties:

$$1. \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

$$2. \mathbf{a} \wedge \mathbf{a} = \mathbf{0}$$

$$3.(\mathbf{a}+\mathbf{b})\wedge\mathbf{c}=(\mathbf{a}\wedge\mathbf{c})+(\mathbf{b}\wedge\mathbf{c})$$

$$4.\,||\mathbf{e}_i\wedge\mathbf{e}_j||=1$$

Definition 2.6. Minkowski product of point sets A and B is set P of all such points that are products of all points $a\in A$ with all points $b\in B$

$$P=A\otimes B=\{a\otimes b, a\in A, b\in B\}\,.$$

Properties of Minkowski product $\otimes$

1. Anti-symmetry $A\otimes B=-B\otimes A$

3. Distributivity $(A\oplus B)\otimes C=(A\otimes C)\oplus(B\otimes C)$

3. Minkowski operations on points and manifolds

Let point p in space E^n be determined as $p=(p_1,p_2,...,p_n)$ and manifold $M\subset E^n$ be given by vector map $\mathbf{r}(u_i)=(x_1(u_i),x_2(u_i),...,x_n(u_i)), u_i\in\Omega\subset R^i$, with coordinate functions $x_j(u_i), j=1,...,n, i\in N$ defined and at least once differentiable on Ω.

Minkowski sum of point p and manifold M is manifold $p\oplus M=M'\subset E^n$ represented by vector map

$$\mathbf{s}(u_i)=(x_1(u_i)+p_1,x_2(u_i)+p_2,...,x_n(u_i)+p_n), u_i\in\Omega\subset R^i$$

Manifold M' is manifold M translated by position vector of point p.

Minkowski product of point p and manifold M is manifold $p\otimes M=M^*\subset E^d$, where $d=n(n-1)/2$, determined by vector map in the form

$$\begin{aligned}\mathbf{p}(u_i)&=(p_1\mathbf{e}_1+...+p_n\mathbf{e}_n)\wedge(x_1(u_i),x_2(u_i),...,x_n(u_i))=\\&=(p_1x_2(u_i)-p_2x_1(u_i),...,p_{n-1}x_n(u_i)-p_nx_{n-1}(u_i)).\end{aligned}$$

Minkowski product M^* of point p and manifold M (curve segment, plane region or surface patch) in space E^3 can be considered as image of manifold M determined by vector map $\mathbf{r}(u_i)=(x_1(u_i),x_2(u_i),x_3(u_i)), i=1,2,3$, under a quasi-central projection from the centre $p=(p_1,p_1,p_1)$ to the plane passing through origin O and perpendicular to position vector of centre p, while

$$\mathbf{r}^*(u_i)=\mathbf{r}(u_i)\cdot T=\mathbf{r}(u_i)\cdot\begin{pmatrix}0 & p_3 & -p_2\\ -p_3 & 0 & p_1\\ p_2 & -p_1 & 0\end{pmatrix}$$

where T is the matrix representing this linear transformation. Image M^* is planar figure located in the plane $p_1x+p_2y+p_3z=0$.

Illustrations of Minkowski product of point and segment of helix, planar disc region and parabolic surface patch are presented in Fig.1.

Special position of point p on a particular coordinate axis yields a quasi-central projection to perpendicular coordinate plane. Thus, three linear transformations

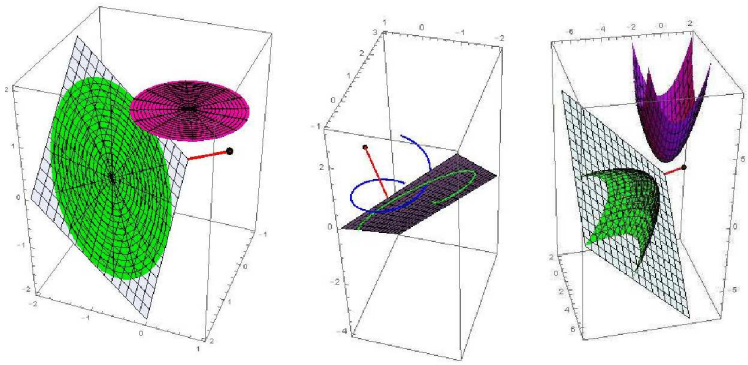

FIGURE 1. Minkowski product of point and various manifolds.

are determined, composed from orthographic projection to respective coordinate plane xy, xz, yz, revolution by angle $-\pi/2$ about origin O in this plane and scaling by nonzero coordinate of the centre of projection p_3, p_2, p_1 with matrices

$$\begin{pmatrix} 0 & p_3 & 0 \\ -p_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -p_2 \\ 0 & 0 & 0 \\ p_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_1 \\ 0 & -p_1 & 0 \end{pmatrix}.$$

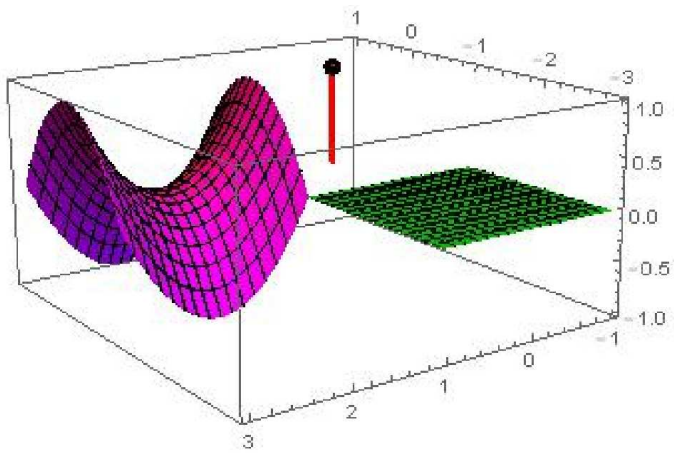

FIGURE 2. Minkowski product of point and surface patch.

In Fig.2 an image of hyperbolic paraboloid is presented under the quasi-central projection from point $p = (0, 0, 1)$ onto plane xy.

Minkowski sum and product of two lines P and Q determined parametrically by vector functions $\mathbf{p}(u) = p + u\mathbf{a}, u \in R, \mathbf{q}(v) = q + v\mathbf{b}, v \in R$ are manifolds

$$S = \mathbf{p}(u) \oplus \mathbf{q}(v) = p \oplus q + u\mathbf{a} + v\mathbf{b}$$

$$R = \mathbf{p}(u) \otimes \mathbf{q}(v) = p \otimes q + q \otimes u\mathbf{a} + p \otimes v\mathbf{b} + u\mathbf{a} \otimes v\mathbf{b}$$

defined for $(u, v) \in R^2$, in form determined by common position of the two lines.

Proposition 3.1. *Let P and Q be parallel lines not passing through reference point at origin. Then their Minkowski sum is a line. If one line is passing through origin, then it is the other line. Minkowski product of the two lines is a plane. If lines form plane passing through origin, then it is a perpendicular line through origin.*

Proposition 3.2. *Let P and Q be intercept lines not passing through reference point at origin. Then their Minkowski sum is a plane. If lines form plane passing through origin, then it is this plane. Minkowski product of the two lines is a hyperbolic paraboloid. If lines form plane passing through origin, then it is a line through origin that is perpendicular to the plane formed by lines.*

Proposition 3.3. *Let P and Q be skew lines not passing through reference point at origin. Then their Minkowski sum is a plane. If one line is passing through origin, then it is plane through the other line. Minkowski product of the two lines is a hyperbolic paraboloid. If one line is passing through origin, then it is a plane through origin perpendicular to the line passing through origin.*

Minkowski sum and product of line P and plane Q parametrically represented by vector functions $\mathbf{p}(t) = p + t\mathbf{a}, t \in R$ and $\mathbf{q}(u, v) = q + u\mathbf{b} + v\mathbf{c}, (u, v) \in R^2$, are manifolds determined for $(u, v) \in R^2$ as

$$S = \mathbf{p}(u) \oplus \mathbf{q}(v) = p \oplus q + t\mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

$$R = \mathbf{p}(u) \otimes \mathbf{q}(v) = p \otimes q + p \otimes (u\mathbf{b} + v\mathbf{c}) + q \otimes t\mathbf{a} + t\mathbf{a} \otimes (u\mathbf{b} + v\mathbf{c})$$

With respect to the mutual position of line and plane, special properties of the resulting manifold can be determined.

Proposition 3.4. *Let P and Q be line and parallel plane, both not passing through reference point at origin. Then their Minkowski sum is a plane parallel to both, while their Minkowski product is a hyperbolic paraboloid. If line is passing through origin, then Minkwski sum is a parallel plane through this line. If plane is passing through origin, then Minkowski sum is the same plane and Minkowski product is plane perpendicular to given line.*

Proposition 3.5. *Let P be line intersecting plane Q, both not passing through reference point at origin. Then their Minkowski sum and product fill the entire space. If just plane is passing through origin, then Minkowski product is 1-parametric system of planes with common line through origin that is perpendicular to given line. If just line is passing through origin, then Minkowski product is plane through origin that is perpendicular to given line. If line and plane share origin, then Minkowski product is plane intersecting given plane in line pasing through origin and perpendicular to given line.*

Minkowski sum and product of two planes P and Q represented by vector maps in forms $\mathbf{p}(u, v) = p + u\mathbf{a} + v\mathbf{b}, (u, v) \in R^2, \mathbf{q}(s, t) = q + s\mathbf{a} + t\mathbf{c}, (s, t) \in R^2$ determine

the whole space, in case they are intersecting and none of them is passing through origin. Minkowski sum of two parallel planes is a plane in the same direction, while their Minkowski product is the whole space.

Proposition 3.6. *Let P and Q be intersecting planes passing through reference point at origin. Then their Minkowski product is 1-parametric system of hyperbolic paraboloids with common line in pierce line of planes through origin. If just one plane is passing through origin, then Minkowski sum is the other plane, and Minkowski product is the whole space formed by system of parallel lines.*

All above propositions can be easily proved by analysing the analytic representations of resulting manifolds, with respect to relations between direction vectors of respective lines and planes. Details and illustrations can be found in [17], [19] and [24].

Consider two curve segments determined parametrically

$$\mathbf{k}(u) = (xk(u), yk(u), zk(u)), u \in K \subset \mathbf{R}$$

$$\mathbf{l}(v) = (xl(v), yl(v), zl(v)), v \in L \subset \mathbf{R}.$$

Minkowski sum of the two curves is a translation surface patch defined on planar region $\Omega = K \times L \subset \mathbf{R}^2$

$$\mathbf{s}(u, v) = (xk(u) + xl(v), yk(u) + yl(v), zk(u) + zl(v))$$

that can be generated by translation of one curve along the other one. Differential characteristics of such surface patch can be represented by means of derivatives of the curve vector maps, abbreviated as

$$\mathbf{k}'(u) = (xk'(u), yk'(u), zk'(u)), u \in K \subset \mathbf{R},$$

$$\mathbf{l}'(v) = (xl'(v), yl'(v), zl'(v)), v \in L \subset \mathbf{R}.$$

First and second fundamental forms of the resulting surface patch and their discriminants can be expressed by the following formulas

$$\phi_1 = \|\mathbf{k}'\|^2 du^2 + 2\mathbf{k}' \cdot \mathbf{l}' du dv + \|\mathbf{l}'\|^2 dv^2, \quad D_1 = (\|\mathbf{k}'\| \cdot \|\mathbf{l}'\|)^2 - (\mathbf{k}' \cdot \mathbf{l}')^2 = \|\mathbf{k}' \times \mathbf{l}'\|^2$$

$$\phi_2 = L du^2 + N dv^2, L = \frac{[\mathbf{k}', \mathbf{l}', \mathbf{k}'']}{\|\mathbf{k}'\| \times \|\mathbf{l}'\|}, N = \frac{[\mathbf{k}', \mathbf{l}', \mathbf{l}'']}{\|\mathbf{k}'\| \times \|\mathbf{l}'\|}$$

$$D_2 = L \cdot N = \frac{D}{D_1}, D = \begin{vmatrix} \|\mathbf{k}'\|^2 & \mathbf{k}' \cdot \mathbf{l}' & \mathbf{k}' \cdot \mathbf{l}'' \\ \mathbf{k}' \cdot \mathbf{l}' & \|\mathbf{l}'\|^2 & \mathbf{l}' \cdot \mathbf{l}'' \\ \mathbf{k}' \cdot \mathbf{k}'' & \mathbf{k}'' \cdot \mathbf{l}' & \mathbf{k}'' \cdot \mathbf{l}'' \end{vmatrix}$$

Gauss curvature is given by formula $K = D \cdot D_1^{-2}$.

Minkowski sum of line segment and circle is a planar region, if they are in one plane or line segment is parallel to the plane of the circle, and it is a circular cylindrical surface patch for non-parallel line segment.

Minkowski product of the two curves is a translation surface patch defined on planar region $\Omega = K \times L \subset \mathbf{R}^2$

$$\mathbf{p}(u,v) = \mathbf{k}(u) \wedge \mathbf{l}(v) = \begin{pmatrix} yk(u)zl(v) - zk(u)yl(v) \\ zk(u)xl(v) - xk(u)zl(v) \\ xk(u)yl(v) - yk(u)xl(v) \end{pmatrix}^T, (u,v) \in \Omega.$$

generated by all such points in the space, whose position vector is vector product of position vector of one point from curve k with position vector of one point from curve l. In case of elementary planar curves some wellknown surface patches can be generated, as e.g. ruled surfaces - cylinders, transition surfaces or conoids, looped strips, torus, and others.
Using properties of vector product, Lagrange identity and following abbreviations

$$\mathbf{k}' \cdot \mathbf{l} = \mathbf{a}, \mathbf{k} \times \mathbf{l}' = \mathbf{b}, \mathbf{k}' \times \mathbf{l}' = \mathbf{c}, \mathbf{k}'' \times \mathbf{l}' = \mathbf{d}, \mathbf{k} \times \mathbf{l}'' = \mathbf{e}$$

differential characteristics of Minkowski product of two curves can be derived in forms

$$\phi_1 = \|\mathbf{a}\|^2 du^2 + 2\mathbf{a} \cdot \mathbf{b} du dv + \|\mathbf{b}\|^2 dv^2, \quad D_1 = \|\mathbf{a} \times \mathbf{b}\|^2$$

$$\phi_2 = D_1^{-1}(L du^2 + 2M du dv + N dv^2), L = [\mathbf{d}, \mathbf{a}, \mathbf{b}], M = [\mathbf{c}, \mathbf{a}, \mathbf{b}], N = [\mathbf{e}, \mathbf{a}, \mathbf{b}]$$

$$D_2 = D_1^{-1}(D_{LN} - D_{M^2})$$

$$D_{LN} = \begin{vmatrix} \mathbf{d} \cdot \mathbf{e} & \mathbf{d} \cdot \mathbf{a} & \mathbf{d} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{e} & \|\mathbf{a}\|^2 & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{e} & \mathbf{a} \cdot \mathbf{b} & \|\mathbf{b}\|^2 \end{vmatrix}, D_{M^2} = \begin{vmatrix} \|\mathbf{c}\|^2 & \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \|\mathbf{a}\|^2 & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{b} & \|\mathbf{b}\|^2 \end{vmatrix}$$

Gauss curvature is given by formula $K = (D_{LN} - D_{M^2}) D_1^{-2}$.

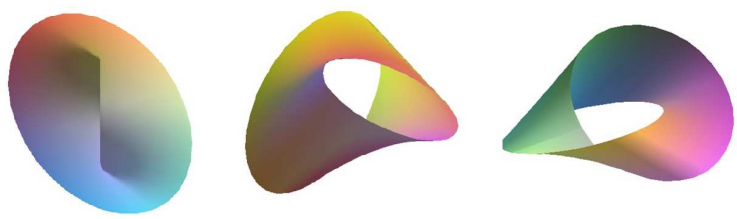

FIGURE 3. Minkowski product of line segment and circle.

Examples of Minkowski products of a line segment and a circle that are located in different super-positions are patches illustrated in Fig.3. For the patch on the left, the line segment is perpendicular to the osculating plane of the circle, line segment and circle are located in the same plane for the transition patch in the middle, and 2-sided strip on the right is the Minkowski product of a line segment parallel to the circle osculating plane.

Other interesting examples are Minkowski sum and product of two circles. Disc is generated as Minkowski sum of circles in parallel planes, while torus is their Minkowski product, as illustrated in Fig.4. Positioning circles into different planes, surface patches of interesting aesthetic forms can be generated. Minkowski sum and product of concentric circles in perpendicular planes result in surface patches presented in Fig.5, sum is on the left, product on the right. Different forms can be achieved with non-concentric circles meeting in one common point, as illustrated in Fig.6.

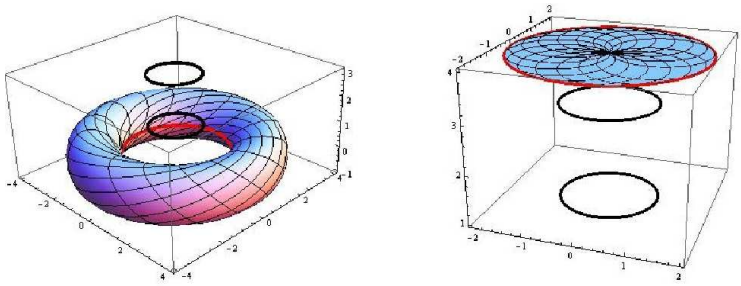

FIGURE 4. Minkowski sum and product of two circles in parallel planes.

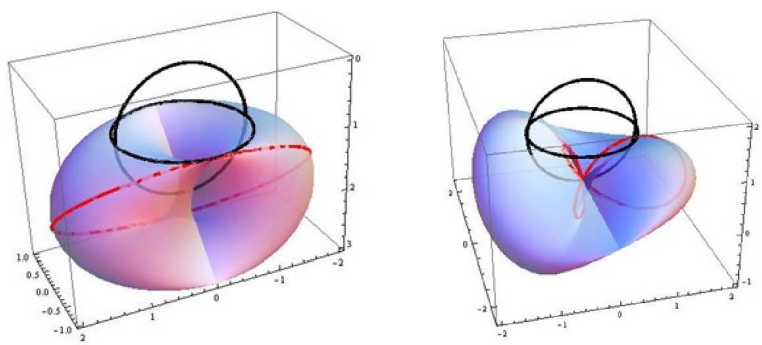

FIGURE 5. Minkowski sum and product of two concentric circles.

Extension of both operations into higher dimensional spaces can be presented on two curve plane segments in $\mathbf{E}^4$ positioned into different planes. Shamrock curve and versiére

$$\mathbf{r}(u) = (a_1 \cos u \ sin^2 u, b_1 \ sinu \ cos^2 u, 0, 0), u \in \langle 0, 2\pi \rangle$$

$$\mathbf{s}(v) = (0, a_2(2v-1), \frac{b_2}{c_2 + d_2(2v-1)^2}, 0), v \in \langle 0, 1 \rangle$$

form Minkowski sum and product, in Fig.7, that are surfaces in different 3-dimensional subspaces of $\mathbf{E}^4$ and $\mathbf{E}^6$ represented by vector maps

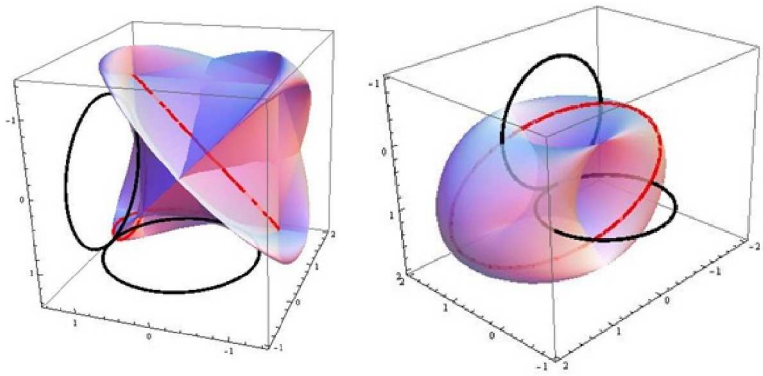

FIGURE 6. Minkowski sum and product of two intersecting circles.

$$\mathbf{s}(u,v) = \begin{pmatrix} a_1 \cos u \ sin^2 u \\ b_1 \ sinu \ cos^2 u + a_2(2v-1) \\ \frac{b_2}{c_2+d_2(2v-1)^2} \\ 0 \end{pmatrix}$$

$$\mathbf{p}(u,v) = \begin{pmatrix} a_1 a_2 (2v-1) \cos u \ sin^2 u \\ \frac{a_1 b_2}{c_2+d_2(2v-1)^2} \cos u \ sin^2 u \\ 0 \\ \frac{b_1 b_2}{c_2+d_2(2v-1)^2} \ sinu \ cos^2 u \\ 0 \\ 0 \end{pmatrix}$$

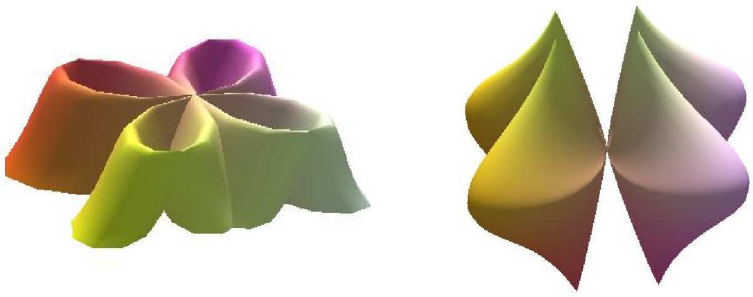

FIGURE 7. Minkowski sum and product of shamrock curve and versiére in $\mathbf{E}^4$.

4. Partial Minkowski operations

Interesting questions can arise in connection to identifying points with equal curvilinear coordinates $u = v$ on surface patches resulting as Minkowski sum or Minkowski product of two curve segments. These point sets can be represented as

so called partial Minkowski operation results. Assuming both operand curves as parameterized for the same parameter on the unit interval $I \subset \mathbf{R}$

$$\mathbf{k}(u) = (xk(u), yk(u), zk(u)), u \in I$$

$$\mathbf{l}(v) = (xl(u), yl(u), zl(u)), u \in I$$

the resulting manifolds are curve segments represented by the following vector maps on $I \subset \mathbf{R}$

$$\mathbf{s}(u) = (xk(u) + xl(u), yk(u) + yl(u), zk(u) + zl(u)),$$

$$\mathbf{p}(u) = \mathbf{k}(u) \wedge \mathbf{l}(u) = \begin{pmatrix} yk(u)zl(u) - zk(u)yl(u) \\ zk(u)xl(u) - xk(u)zl(u) \\ xk(u)yl(u) - yk(u)xl(u) \end{pmatrix}^T,$$

which are located on the respective surface patches, the Minkowski sum or the Minkowski product of differently parameterized original curve segments.

Examples of curves generated as partial Minkowski product of two equally parameterised circles in different position and sharing none, one or two common points are given in Fig.8. The same curves are illustrated also in Fig.4, Fig.5 and Fig.6, as curves located on respective surface patches that are generated by Minkowski operations on the same circles, but differently parameterised. Partial Minkowski sum of two equally parameterised circles is a circle for all positions of the two operands.

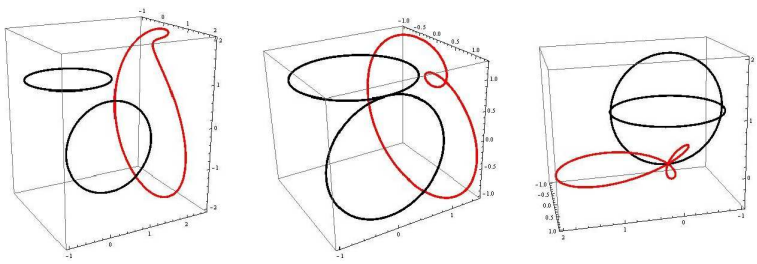

FIGURE 8. Partial Minkowski product of two circles.

Generalisation of Minkowski point set operations are Minkowski point set combinations - summative and multiplicative, leading to more powerful modelling tools enabling determination of two-parametric families of surfaces or curves with extreme elasticity of form. Shape modifications are depending on values of two real parameters in the role of scaling coefficients of the two operand sets.

A two-parametric family of curves named "Laces" can be introduced and investigated, while particular representatives of this family are generated by means of Minkowski summative or Minkowski multiplicative combinations of two euqally parameterised curve segments in the Euclidean space $\mathbf{E}^3$.

Let two curve segments K and L be determined by respective vector maps defined on the same interval $I \subset \mathbf{R}$.

Definition 4.1. Minkowski summative combination of curves K and L is a two-parametric family of curve segments in $\mathbf{E}^3$

$$S = a.K \oplus b.L, a, b \in \mathbf{R}$$

parametrically represented on $I \subset \mathbf{R}$ by vector maps

$$S : \mathbf{s}(u) = a.\mathbf{k}(u) + b.\mathbf{l}(u) = (xs_1(u), xs_2(u), xs_3(u)),$$

where the following relations hold for coordinate functions

$$xs_i(u) = a.xk_i(u) + b.xl_i(u), i = 1, 2, 3.$$

Definition 4.2. Minkowski multiplicative combination of curves K and L

$$P = a.K \otimes b.L, a, b \in \mathbf{R}$$

is a family of curve segments in $\mathbf{E}^3$, parametrically represented on $I \subset \mathbf{R}$ by vector maps

$$P : \mathbf{p}(u) = a.\mathbf{rk}(u) \wedge b.\mathbf{rl}(u) = (xp_1(u), xp_2(u), xp_3(u)),$$

where the following relations hold for coordinate functions

$$xp_k(u) = ab(xk_i(u)xl_j(u) - xk_j(u)xl_i(u)),$$

for all combinations of pairs of coefficients $i, j = 1, 2, 3, i \neq j$, while $k = 1, 2, 3$.

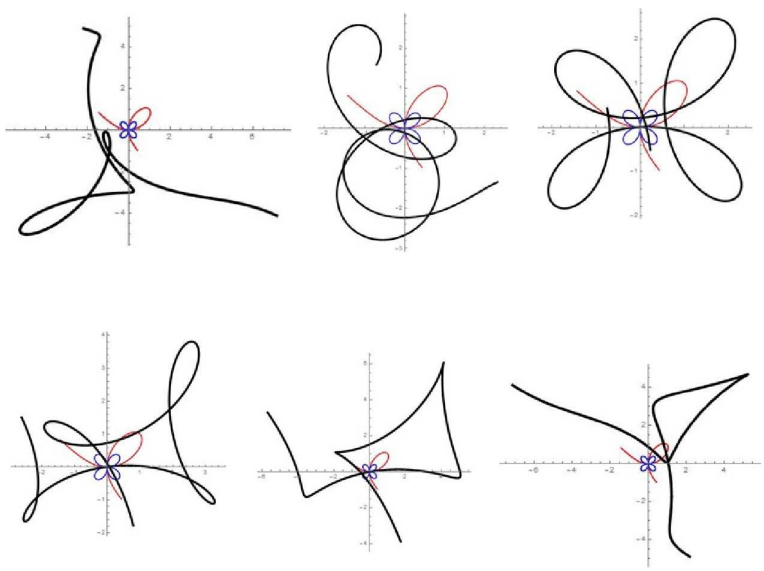

FIGURE 9. Minkowski summative laces.

Examples of family of laces generated as Minkowski summative combinations of equally parameterised shamrock curve and leaf of Dessargues in one plane are in Fig. 9, while some forms of Minkowski multiplicative combinations of this two curve segments located in different planes are presented in Fig.10.

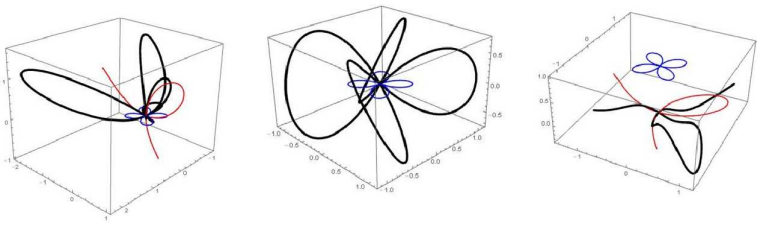

FIGURE 10. Minkowski multiplicative laces.

CONCLUSION

Minkowski point set operations and their generalisations presented in this paper can be used for many applications in various fields of computer graphics, geometric modelling, design, architecture and art. These modelling algorithms serve for modelling various forms of curves and surfaces, in morphing deformation of curve shape, or adjusting the shape C to a given desirable form A, while $C = k.A \oplus B$. Decomposition of point set C to set components can be achieved as Minkowski difference of point set C and k-multiples of point set A, which leads for known C and A to $B = C \oplus (-k.A)$.

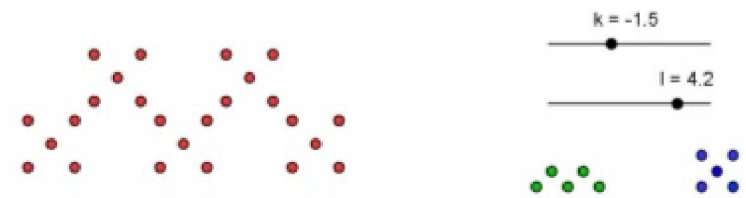

FIGURE 11. Partial Minkowski summative combinations used for shape adjustment.

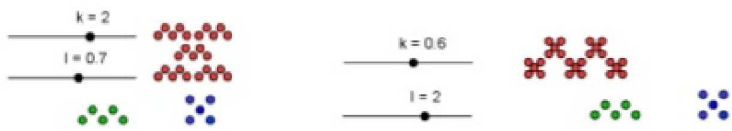

FIGURE 12. Minkowski summative mosaics generated from dicrete point sets.

In Fig.11, on the left, curve segment C, Minkowski summative combination of ellipse B and k-multiples of curve segment A, is adjusted to take the shape of the curve segment A, while on the right, where curve segment C is Minkowski summative combination of circle B and k-multiples of curve segment A, it is adjusted to take the shape of circle B.

Among many other interesting applications of Minkowski point set combinations, Minkowski summative combinations of discrete point sets can be mentioned, which

can be easily used to generate various point mosaics. Two sets A and B, each consisting from 5 different points located in one plane, are combined by means of Minkowski summative combination $C = k.A \oplus l.B, k, l \in \mathbf{R}$, forming thus a family of mosaics with various shapes that can be easily modified by chosen values of real parameters k and l. Some illustration are included in Fig.12.

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