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ADVANCED ELEMENTARY GEOMETRY - A RESEARCH PLAY GROUND FOR YOUNG AND OLD

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ABSTRACT. "Elementary Geometry" received its name from Euclid's "Elements", it deals with "elements", i.e. points, lines, circles, planes and "relations", i.e. incidences, proportions, lengths and angles. While in former times Elementary Geometry constituted a fixed part in mathematics syllabuses, providing a trainings field for logic reasoning and coordinate free proving methods, the topic now has little scientific esteem and seems to vanish completely in Maths education. The paper tries to show that Elementary Geometry still has its merits and even opens up for new fields in Mathematics. Connecting Elementary Geometry with Projective Geometry and Circle and Chain Geometry extends it to "Advanced Elementary Geometry". These extensions sometimes give better insight into the nature of a basic theorem and it justifies the preoccupation with that subject. The paper aims at providing teachers with perhaps fascinating geometric problems. Most of the presented material stems from other publications of the author and is nothing but a "closer look" to very basic elementary geometric theorems. Thereby iteration and generalization as the standard methods of research in mathematics are applied and often lead to astonishing facts and incidences. Modern dynamic visualization tools together with automatized theorem proving software quickly allow to formulate statements and theorems and therefore are a great stimulus to do research at an early stage of mathematics education.

1. INTRODUCTION [1]

Due to Euclid's 13 books with the title "The Elements" we subsume geometric problems dealing with points, lines, planes and circles and spheres and their mutual relations in the Euclidean plane and 3-space as "Elementary Geometry". This topic dates back to at least the 5^{th} century B.C. and it was the greatest stimulus for developing most parts of nowadays geometry and mathematics. For engineers and architects it still provides abstractions of their objects as sets of "primitives". From elementary school onwards it provides pupils and students with challenging examples, where they can gain competence in logic reasoning and coordinate free "synthetic" proving methods.

All these merits cannot avoid that, among "hardcore mathematicians", elementary geometry is sort of a topic 'non grata'. But is this justified? Is Elementary Geometry just nothing but an 'evergreen' for retired oldies? Modern mathematics education worldwide, also at Universities, is geometry free with few exceptions. What nowadays people, including Maths professors at Universities, usually get and

Key words and phrases. Elementary Geometry, Euclidean Geometry, Affine Geometry, Projective Geometry, Minkowski Geometry.

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remember of Elementary Geometry is hardly more than the theorems of Pythagoras and Thales. And this is indeed not mainstream mathematics!

Would it help to add "Advanced" or "Computational" to it to raise its image as is proposed in [43]? Via Dynamic Geometry Software one can make interesting discoveries by trial and error or just by chance. For some people (e.g. Hirotaka Ebisui, who found more than 4000 "new theorems", publishing some of them via Facebook or telling them just to some few friends) this researching method is a starting point for aim-oriented research, too. The proof of the results (or better: guesses) found that way often are very tricky or they demand deep and broad knowledge in all sort of Geometry. The next chapters will show, by presenting some recent examples, that Elementary Geometry still stimulates research in other mathematical fields and gives a hint for better understanding of often little reflected facts in Geometry and Mathematics.

Typical questions arising from an arbitrary well-known elementary geometric fact are:

- Where does the fact belong to: to Euclidean, Affine or Projective Geometry?
- Does the (synthetic) proof relay on special closure conditions (Desargues, Pappus-Pascal, Fano, Miquel)?
- Is this fact valid for other geometries, e.g. non-Euclidean or Circle and Chain Geometries?
- Are reals the necessary coordinate field or can they be replaced by other fields, rings, etc.?
- Are there generalisations to higher dimensions? How to adapt such generalisations?
- What about combinatorial aspects, when enriching the start figure with additional elements?
- Is it possible to iterate the basic construction/definition of the given basic object?
- Can the facts be grouped to other similar facts?

Besides all that the astonishing beauty of the unexpected incidences within elementary geometric figures is already enough to become addicted to that kind of "pure geometry", E.g. [19] can act as a beautiful introduction to the wonders of Geometry and Mathematics.

1.1. **Reasons which make the low value of Elementary Geometry understandable.**

There is now "good" systematic for the thousands of theorems and statements. To improve this would need sort of a "meta-systematic". Good attempts of a systematic treatment of e.g. "triangle geometry" are [14], [15], [17], [2], [3], [32], [12], [22], [6]. In [3] one finds a good systematic treatment of tetrahedra, too. But one could add quite a lot of additional viewpoints and different ways to generalize the presented material. So even we would like to find a systematic approach a la Linn's classification system of plants we just end up with a chaotic wickerwork of relations between theorems based on the usually very fruitful principle of generalization. Recent examples are e.g. [20] on the Theorem of Thales or [45] on the Golden Mean.

Synthetic proofs are tricky, and analytic proofs sometimes hide the essence of the fact. (Here belong proofs of geometric extreme value problems "without differentiation".)

Among the people making findings in Elementary Geometry there are many hobby mathematicians with no broader mathematical education. Many of their "findings" become trivial when putting them into the context of Projective Geometry or of geometries over rings. Modern dynamic graphics software and automatic theorem proving software makes it easy to state such new findings, but these software products give no hint about the context and the perhaps more general geometry the found 'new theorem' belongs to.

Modern mathematics education worldwide, also at Universities, is geometry free with few exceptions. What nowadays people, including Maths professors at Universities, usually get and remember of Elementary Geometry is hardly more than the theorems of Pythagoras and Thales. And this is indeed not mainstream mathematics!

1.2. **Positive aspects of Advanced Elementary Geometry.**

During history there have been very famous mathematicians among Elementary Geometry researchers, e.g. L. Euler, C.F. Gau, B. Pascal and A. Mbius, just to mention a few "old" ones. With L. Euler we connect the Euler line of a triangle, a typical concept of Euclidean geometry. C.F. Gau found the Gau line of a quadrilateral, a concept of Affine Geometry. A projective geometric concept due to B. Pascal is the Pascal axis of a hexagon inscribed into a conic. A. Mbius and his "inversion" lead to Euclidean Circle Geometries, c.f. [7], [8]. The list of important researchers could of course be extended up to recent days and Kimberlings e-book resp. list of remarkable triangle points reads somehow as a who-is-who of (Advanced) Elementary Geometry researchers.

The "Foundations of (Projective) Geometry", once a flourishing topic in the 1960ies and 70ies of the last century, where nourished by transforming geometric configurations into algebraic statements. Another classical topic, the geometry of (semi-) regular polyhedra is the basis for research in n-polytopes, n-simplices and, to a certain extent, to Convex Geometry and Combinatorial Geometry (see e.g. [9], [48]) and Rigidity ([1], [34], [36], [37], [39]), while another trace of generalizations lead to tilings in non-Euclidean spaces (see e.g. [30]).

Even (elementary) Number Theory can be connected with b)! The number representations 2.5 and 5/2 are of equal value and show the same integers. St. Deschauer (see [42] and [11]) asked for all numbers p/q with this property. This leads to a Diophantine problem of integer solutions of quadratic equations, which mean hyperbolic paraboloids. Considering solutions being reals, then the periodic decimal number $1,1\dot{1}$ (for $p = 10, q = 1$) belongs to here, too. But also V. Spinadel's "Metallic Means" [35] being the positive solutions of quadratic equations of type

$$x^2 + px - q = 0, \ (p, q \in \mathbb{R}), \tag{1}$$

are related to Deschauer's problem. Higher order generalisations of (1) are the cubic numbers of van der Laan and L. Rosenbusch (see [49]) with relevance in Architecture.

We associate cubic problems with "Mathematical Origami", a now incredibly fast growing research topic with interesting technical applications, see [10]. It is nourished by important elementary mathematical problems, as there are the "trisection of angles", the "doubling of a cube" and the "construction of a regular heptagon". As Origami deals with reflections, it is also connected with F. Bachmann's fundamental idea of "building geometries based on the concept of reflections" [5], and this point of view opens up new research grounds of theoretical origami, as e.g. non-Euclidean and higher-dimensional origami.

Another fast growing topic is "Minkowski Geometry", the geometry of metric spaces, see [38] and [Martini] and his group. Most of the statements read as follows: "A metric plane (space), where a statement valid for Euclidean planes (spaces) holds, is Euclidean!" But there are some few Euclidean theorems, which also hold in all or at least some Minkowski planes (spaces). A nice example of this type is e.g. the "beer mat theorem" [4]. Even spirals can be considered, see [46].

In the following we treat some few examples belonging to the mentioned extensions of Elementary Geometry. The obtained results can be arranged in several groups:

(a) "Surprising discoveries" in 2D- or 3D-Euclidean Geometry occurring by chance. As an example I want to mention the *Pavillet-tetrahedron* to a given triangle, see [31].
(b) Euclidean generalisations of discoveries of type (a). E.g. the van der Laan number generalizes the Golden Mean and it lead to further generalisations, see [35].
(c) Problems of "Intuitive Geometry". Here belong the many "20$-questions" posed by P. Erds.
(d) Modifications of the results in (a) and (b) for n-spaces (over general fields) or for non-Euclidean geometries or for Circle Geometries (over general rings). For example, the "Theorem of Miquel" originally was formulated as a statement for elementary triangle geometry, but it turned out to be of essential meaning for the algebra of circle geometries, see [7].
(e) Modifications of classical Euclidean statements for general (n-dimensional) metric spaces, so-called Minkowski spaces, see e.g. [38], [24], [25].

The following examples refer to these aspects. Of course they are just curiosities, but with the aim to make pupils, students and their teachers curious and give them self-confidence to do research by their own. Usual maths teaching provides self-contained "ready made material" leaving little chance for own creativity. This paper wants to present a way to "teach creativity" by posing questions starting from a few very well-known statements.

2. The Theorem of Pythagoras - a simple "Lego"

The Theorem of Pythagoras is general knowledge of the ordinary Joe. There are many interesting proofs of it, but nobody seems to "play" with it. For example, the analytic expression

$$a^2 + b^2 = c^2 \tag{2}$$

is an algorithm summing two numbers to receive a third one. Interpreting this algorithm as a recursive one, one can receive the set of natural numbers as well as the set of Fibonacci numbers, see Figure 1 and [47].

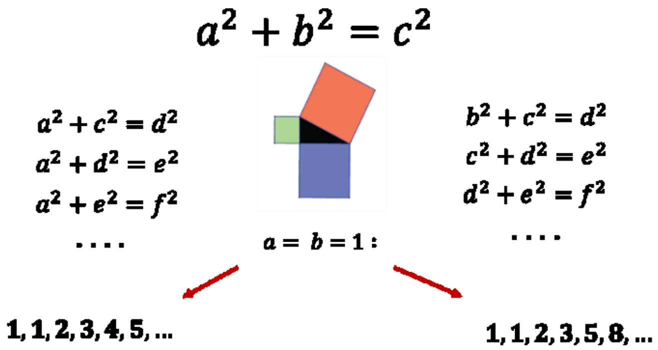

FIGURE 1. Iterative recursion algorithm connected with Pythagoras' Theorem

For the left column in Figure 1 there exists the well-known visualisation by the "root spiral". The right column leads to a visualisation of the Fibonacci sequence by a "Fibonacci root spiral", too, see Figure 2. With respect to the side ratio of the limit triangle we also receive a "Golden Pythagoras Spiral", too, see Figure 3. So the usual visualization by squares and golden rectangles is not the only possibility! Furthermore, if we refrain from the orthogonality of the Pythagorean triangles and just use the ratio of side lengths, this visualization method can also be performed in non-Euclidean planes, as e.g. the hyperbolic plane, see [46].

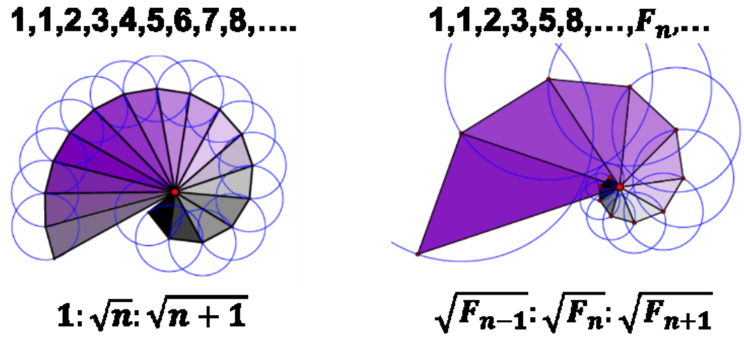

FIGURE 2. Root spiral (left) and Fibonacci root spiral (right)

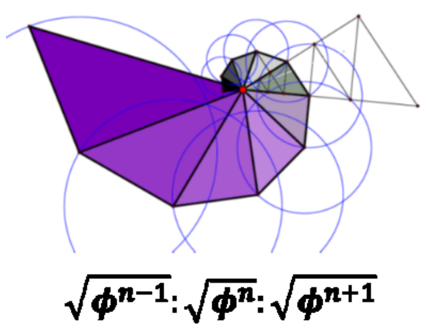

FIGURE 3. "Golden Pythagoras Spiral"

A discovery of H. Ebisui [13] added to a Pythagorean triangle at first the squares over the sides as usual. The convex hull of this standard figure is a hexagon and over the sides connecting vertices of the first squares he erected again squares and iterated this process, see Figure 4, (an original figure of H. Ebisui).

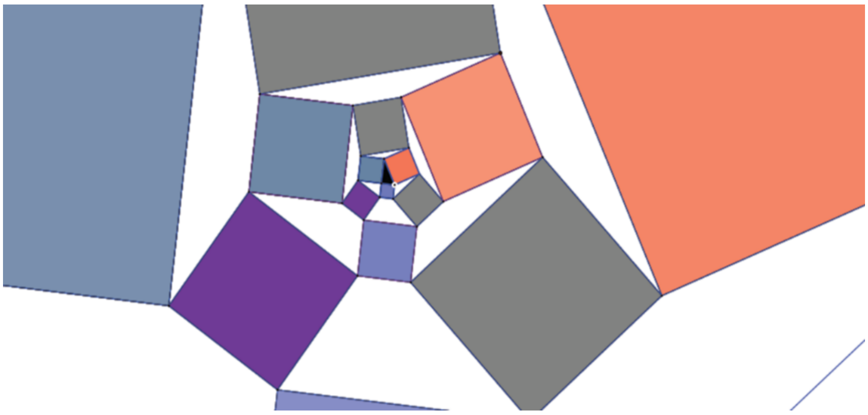

FIGURE 4. H. Ebisui's iteration of the standard visualization of Pythagoras theorem

Analyzing Figure 4 he formulated the following statements:

1st Theorem of Pythagoras-Ebisui: *At each stage yields "blue"+"blue" = "rose".*

2nd Theorem of Pythagoras-Ebisui: *At each stage yield "grey"+"grey" = 5 x "violet".*

3rd Theorem of Pythagoras-Ebisui: *(a) At any stages the intersections of the first diagonals are collinear on lines orthogonal to the circum-hexagon's sides. (b) All those 6 lines pass through one point, which is the gravicentre of the start triangle (Figure 5 at left). (c) All this is valid also for general start triangles (Figure 5 at right).*

The easy proofs interpret the Euclidean plane as Gau-plane and use complex numbers.

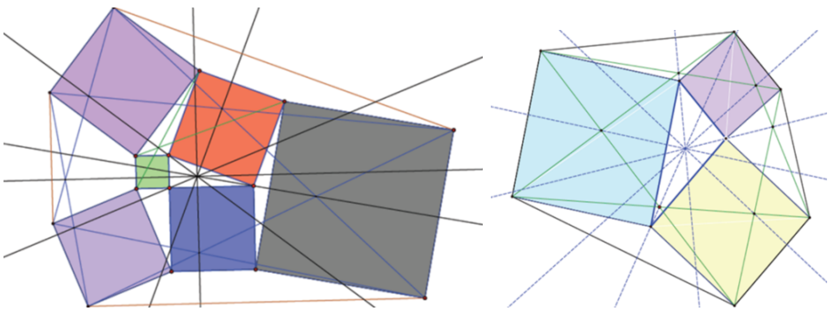

FIGURE 5. Ebisui's Pythagoras extensions formulated in the 3rd Theorem of Pythagoras-Ebisui

Figure 5 (left) reminds on "chains of squares" and there are again many theorems, which are easy to proof and well suited as geometry/mathematics problems for high school pupils, see Figure 6.

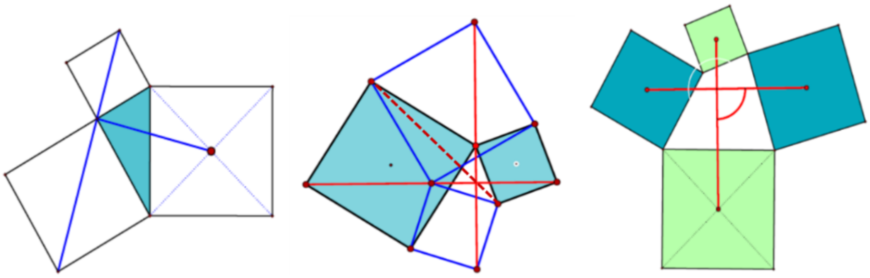

FIGURE 6. Theorems about "closed chains of squares": a) orthogonal segments occurring at a right angled triangle, b) orthogonal segments, start figure for "fractal iteration", c) orthogonal segments of same length occurring at a chain of four squares.

3. The Fermat-Steiner point of a triangle and generalizations

As there are no squares in hyperbolic and elliptic planes the findings concerning chains of squares and visualized in Figure 4, Figure 5 and Figure 6 have no meaning in these planes. Therefore one might have the idea to replace squares by equilateral triangles, which exist in all Cayley-Klein planes. In the following figures we start with the well-known

Theorem of Fermat-Steiner: *Adding equilateral triangles outward to the sides of an arbitrarily given triangle Δ in the Euclidean plane results in a new triangle, which is in perspective position with the given triangle. The perspectivity center is called Fermat point F of the given triangle Δ and, in case of the greatest angle of the given triangle does not exceed 120, it minimizes the distance sum to the vertices of Δ. The segments connecting the vertices of the new triangle with the opposite vertices of Δ intersect by 120 and have equal lengths.*

Adding again equilateral triangles to the new triangle, see Figure 7 is the start of an iteration process. It turns out that at each stage the Fermat point F and the perspectivity lines are fixed and the limit ratio of side lengths of the triangles is 1:1:1. The iteration process could therefore be called a "regularizing process".

A first generalization could change the place of action from Euclidean to hyperbolic or elliptic planes. But here it turns out that the extension of the Fermat-Steiner property is not true!

The question arises, whether extensions to three and higher dimensional Euclidean spaces are possible. To the faces of an arbitrarily given tetrahedron Δ one could add equifaced tetrahedra to the face triangles of Δ. Another possibility would be to inscribe maximal equilateral triangles to the faces of Δ and use them as bases for added regular tetrahedra. This topic seems not to be treated yet!

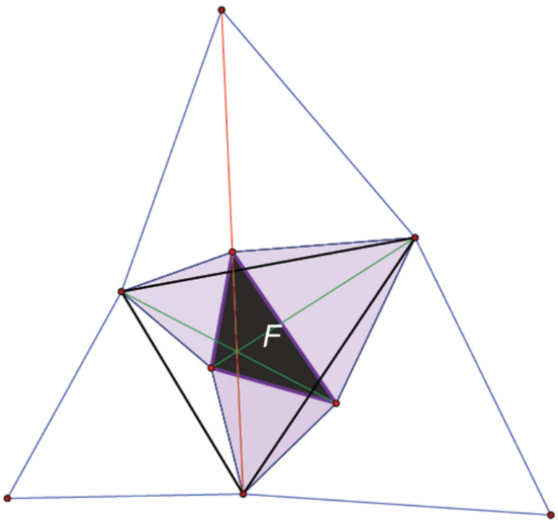

FIGURE 7. A set of triangle chains and their common Fermat-Steiner point

4. Elementary geometric extremal problems seemingly solved without differentiation

The Fermat-Steiner theorem (Chapter 3) formulates an extreme value problem, which is solved "without differentiation" just by applying the theorem of the angle at circumference: F is the common intersection point of the circum-circles of the equilateral triangles added to the given triangle Δ.

A similar problem is the following one: *Find the triangle Δ' inscribed to an arbitrarily given acute triangle Δ such that its circumference is minimal.* The well-known geometric solution of this three-variate problem uses reflections and the fact that an isosceles triangle with fixed top vertex angle has minimal basis if its legs are minimal. The path to the solution Δ', which is just the pedal point triangle of the altitudes of Δ is shown in the set of figures in Figure 8.

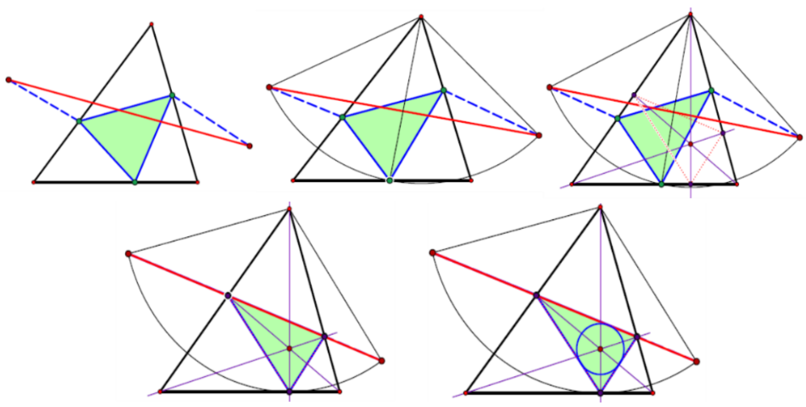

FIGURE 8. Visualization of the geometric proof of the "minimal inscribed triangle" to a triangle in the Euclidean plane

The same geometric proof holds also on the sphere (Figure 9) and thus in the elliptic plane!

For a mathematical treatment differentiation is inevitable. In the geometric treatment differentiation hides in the already solved problem of finding geodesic lines in those spaces. In the Euclidean plane geodesics are straight lines, on the sphere they are great circles.

Generalizations to higher dimensional Euclidean and Cayley-Klein spaces seem to be still an open research topic.

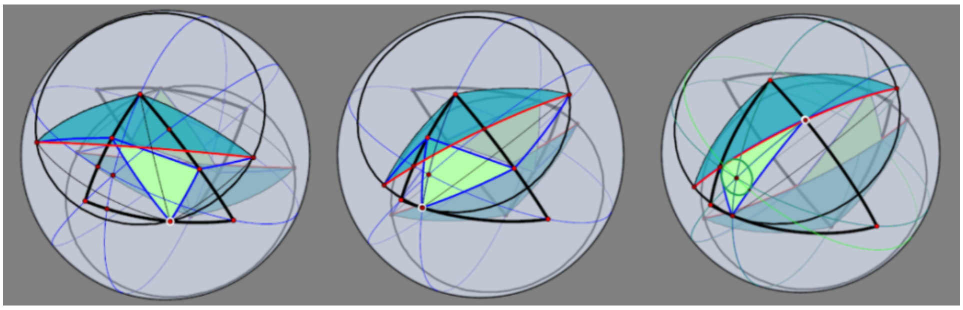

FIGURE 9. Visualization of the proof for the spherical triangle with minimal circumference inscribed to a spherical triangle

5. Congruences and similarities

The congruence theorems for triangles and the intercept theorem are standard topics in primary and secondary school education. The concept "transformation group" is introduced at high school level. But constructions of fixed elements of a congruence or similarity are not standard topic of education and hardly are thought at universities, even such constructions are very simple and do not use more than the theorem of the angle at circumference.

Figure 10 visualizes the construction of fixed elements of an indirect similarity (which is a pair of orthogonal lines) and an indirect congruence (which is also called "slide reflection").

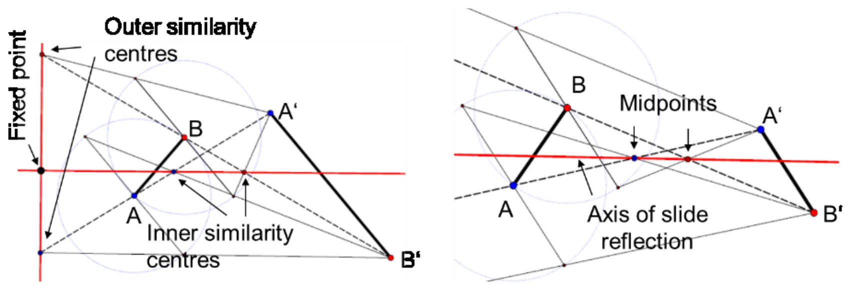

FIGURE 10. Construction of fixed elements of an indirect similarity (left) and a slide reflection (right)

Constructions of fixed elements of congruences in an elliptic or hyperbolic plane are not topics of school education and involve Projective Geometry.

Direct congruences connect Elementary Geometry with Kinematics. In this paper we omit further discussions of this extremely wide topic.

6. R. Goormaghtigh's "orthopole mapping"

R. Goormaghtigh projected the vertices of a fixed triangle $\Delta = UVW$ orthogonally onto an arbitrarily chosen line g and (see [33] and Figure 11) and showed, that the parallels to the altitudes of Δ through the image points A, B, C of U, V, W meet in one point G, which he called the "orthopole" of g. This orthopole mapping is quadratic: If g rotates around a point P, then the orthopole G runs through a conic p, the image of P.

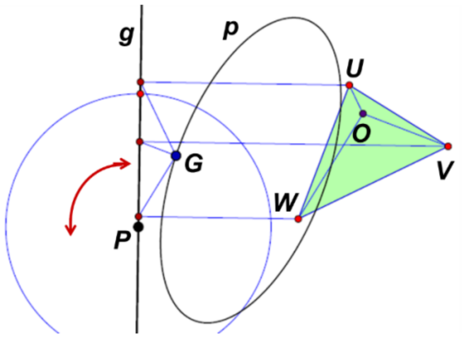

Figure 11. R. Goormaghtigh's "orthopole mapping" $\gamma : g \mapsto G$ and its reverse $\gamma^(-1) : P \mapsto p$

A first extension of this interesting closure property due to Goormaghtigh states, that if Δ' is a triangle perspective affine to Δ with affinity axis g, then the parallels to the altitudes of Δ through the vertices of Δ' intersect in a point O' and this point is incident with the line connecting the orthopole G of g with the orthocentre O of Δ, see Figure 12.

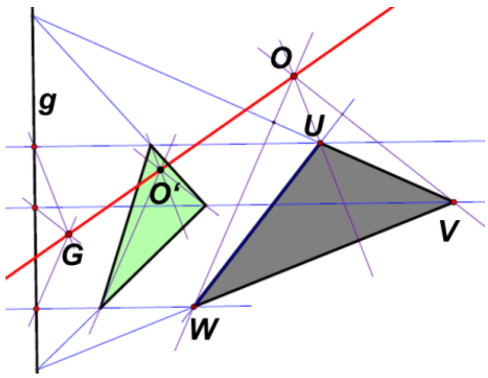

Figure 12. R. Goormaghtigh's extension of the orthopole mapping

A second extension starts with a quadrilateral and projects its partial triangles orthogonally onto the respective fourth side, see Figure 13.

Result: *The orthopoles of the four sides of a quadrilateral with respect to the respective remaining partial triangle are collinear with the line connecting the orthocentres of these four partial triangles.*

Remark: This extraordinary line contains also the two (in algebraic sense) so-called Bodenmiller points of Δ. Here theorems of Bodenmiller (see e.g. [16]) and Wallace-Simson (see e.g. [14], [15] and e.g. [18]) interfere and challenge to a closer look.

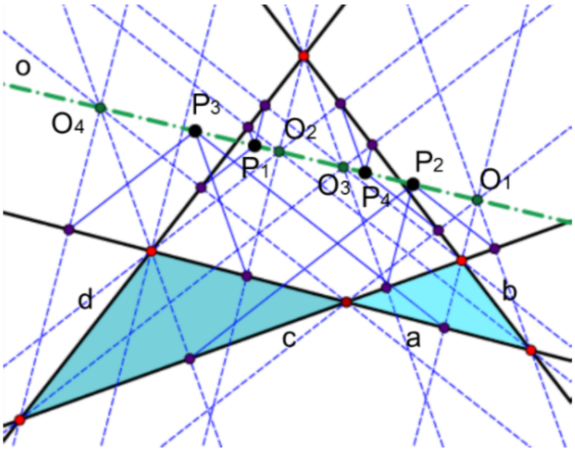

FIGURE 13. The orthopoles of the sides of a quadrilateral with respect to the remaining triangles are collinear

Spatial analogues of the orthopole mapping seem to be not considered up to now. E.g. an arbitrarily given tetrahedron Δ having four altitudes, which either belong to a regulus Σ or intersect in pairs or are concurrent (c.f. [21]), could be projected normally onto a plane. So the question arises, whether the four parallels to the altitudes through the image points of the vertices of Δ again span a regulus Σ' (or its degenerated versions) or not.

A somehow related problem seems to be that of A. Pavillet's discovery, see [31] and Figure 14: *Start with a triangle $\Delta' = ABC$ and its in-centre I. Take A, B, C as the orthogonal projections of three vertices P, Q, R of a tetrahedron Δ and its fourth vertex be I. Let the distances $\overline{PA}, \ldots$ be the same as those of A, to the touching points of the in-circle with Δ', then the tetrahedron $\Delta = PQRI$ is orthocentric.*

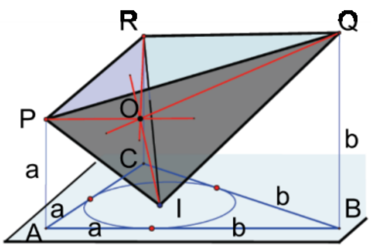

FIGURE 14. The Pavillet tetrahedron to a given triangle is orthocentric

Also this statement can be seen as starting point for many interesting questions, e.g. what about the reverse problem, i.e. starting with an orthocentric tetrahedron and look for orthogonal projections, such that the images of the vertices fall onto the vertices of a triangle and its in- resp. one out-centre? What about higher-dimensional generalizations?

7. An incidence theorem of H. Ebisui and a remarkable correlation

In an e-mail to me H. Ebisui has sent the following Figure 15 (left), again with his standard comment: "I found a new theorem, please enjoy!" Take a triangle $\Delta = ABS$ and its circumcircle c and intersect c and the sides $AS = r$, $BS = t$ with an arbitrarily given line s. Project the intersection points with c from A and B onto t resp. r and those with r and t onto c as indicated in Figure 15 (left). The second pair of projection lines intersect in point X and the diagonals of the occurring quadrangles intersect in Y. Then the points X, Y, S are collinear with say an "Ebisui line".

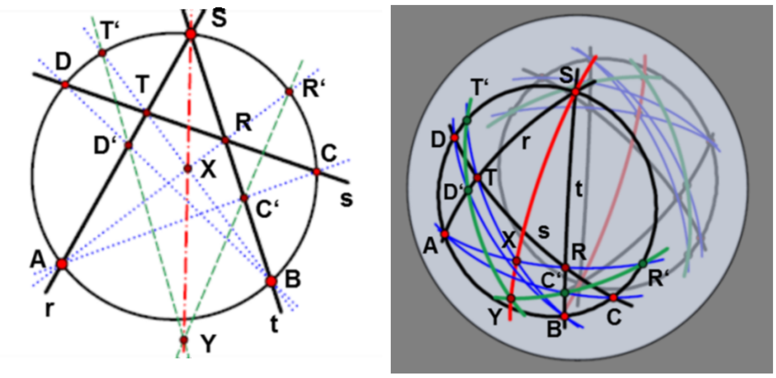

Figure 15. H. Ebisui's discovery of collinear points (left) and its spherical version (right)

It turns out that this incidence theorem belongs to Projective Geometry and is a theorem about special pencils of conics. It is therefore valid in all Cayley-Klein planes, as e.g. visualized on the sphere in Figure 15 (right).

The question immediately arising from Ebisui's discovery is: What about the line $AB =: u$ and applying his construction three times?

Considering the quadrilateral $\Lambda = rstu$ we find that the three occurring Ebisui lines are just its diagonals. The construction shows that the diagonal triangle of Λ is in perspective position with each of the partial triangles of Λ, see Figure 16.

Result: *The partial triangles of a quadrilateral $\Lambda = rstu$ are in perspective position with its diagonal triangle $\Delta = XYZ$. The four perspectivity centres R, S, T, U and the respective sides r, s, t, u uniquely define projective correlation $\pi : \{points\} \to \{lines\}$, which turns out to be involutoric, i.e. π is a canonically defined polarity to the given quadrilateral.*

Remark: We might also consider another projective correlation based on Goormaghtigh's orthopole mapping (Chapter 6), which is defined by mapping the

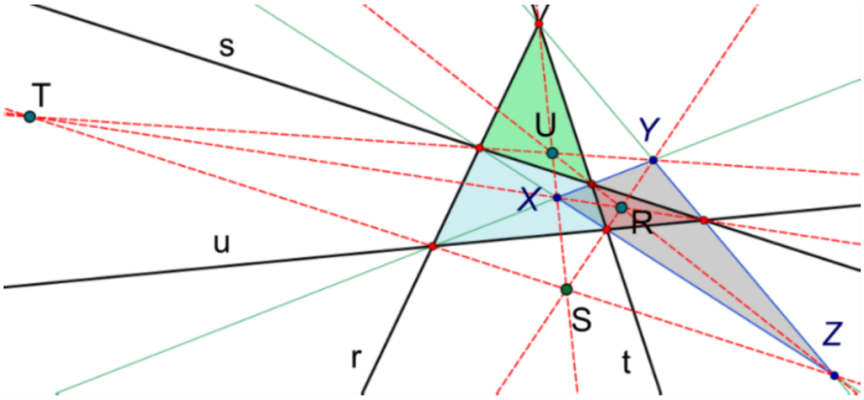

FIGURE 16. The partial triangles of a quadrilateral are in perspective position with its diagonal triangle

sides of a quadrilateral $\Lambda = rstu$ to their orthopoles with respect to the diagonal triangle $\Delta = XYZ$ of Λ. Note that this correlation involves Euclidean structure.

8. PERSPECTIVE TRIANGLES AND DESARGUES' $(10_3, 10_3)$−CONFIGURATION

The occurrence of perspective triangles might lead to revisit the classical and well-known Theorem of Desargues, see [44]. Thereby one should distinguish between a labelled figure of perspective triangles and the (un-labelled) Desargues' $(10_3, 10_3)$−configuration. While the first point of view defines a uniquely defined homology with a characteristic cross-ratio, the configuration point of view has to consider 10 such homologies, which must be dependent. That will say that their characteristic cross-ratios d_i must fulfil conditions.

Remark: While the cross-ratio of ordered quadruplets is a uniquely defined number, not ordered quadruplets, "as a geometer sees them", give rise to the six cross-ratio values,

$$c, \frac{1}{c}, 1-c, \frac{1}{1-c}, 1-\frac{1}{c}, \frac{c}{c-1}, \tag{3}$$

which are valid simultaneously.
Therewith, using suitably selected values d_i from the six possible ones, one finds the conditions (see [44])

$$\begin{array}{lllll} d_3d_5/d_7 = -1, & d_5d_8/d_1 = -1, & d_2d_9/d_8 = -1, & d_2d_3d_4 = -1, & d_4d_6/d_7 = -1, \\ d_6d_9/d_1 = -1, & d_3d_8/d_{10} = -1, & d_2d_5/d_6 = -1, & d_7d_{10}/d_1 = -1, & d_4d_{10}/d_9 = -1, \end{array}$$

and

$$d_2d_5d_9 = d_3d_7d_8 = d_4d_6d_{10} = d_3d_5d_{10} = d_1, \tag{4}$$

from what follows

$$\prod_{i=1}^{10} d_i = d_1^4. \tag{5}$$

If perspective triangles allow a re-labelling such that they again are in perspective position, we call them "multi-perspective". Figure 17 shows cases of triangles which are in exactly two, three and even four perspective positions.

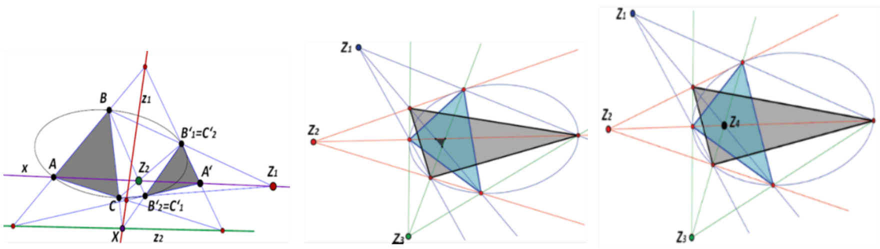

FIGURE 17. Multi-perspective triangles.

An interesting special case form so-called "Golden Hexagons" (see [40]). They consist of six points positioned such that any partition into two triangles allows a labelling such that they are in perspective position, see Figure 18. In this figure occur golden ratios and "golden cross-ratios". Note that point quadruplets with the golden cross-ratio value ϕ are, besides harmonic quadruplets, the only ones with less than 6 (real) cross-ratio values (3). Among all real quadruplets of points the harmonic and golden ones are therefore projectively distinguished and do not afford affine or Euclidean structure.

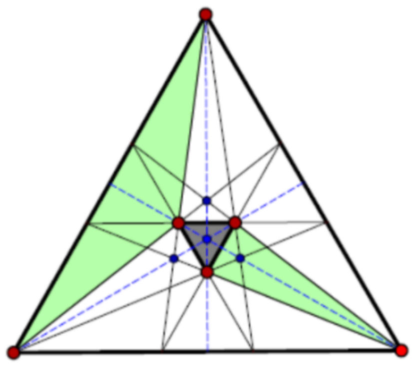

FIGURE 18. "Golden Hexagon"

9. Elementary Geometry with circles: more Ebisui curiosities

One of the many discoveries of H. Ebisui concerns two circles a, b, which intersect in two points S and T: Project S from centre A of a onto b and receive Q, project S from centre B of b onto a and receive P. Intersect the line PQ with a, b and get points X and Y, see Figure 19 (left).

Ebisui's two circle theorem:

(6) $$\overline{SX} = \overline{SY} = \overline{ST}$$

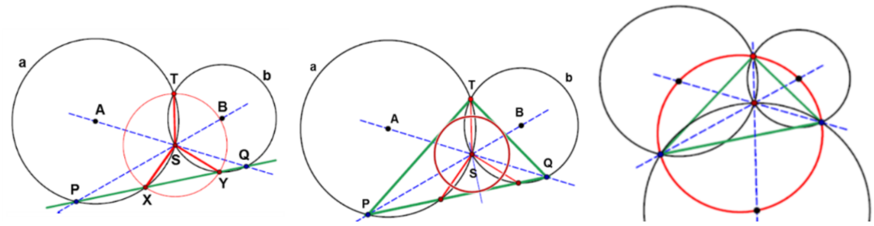

FIGURE 19. Ebisui's two circle theorem and extensions

The proof uses reflections and the theorem of the angle at circumference. This discovery can be extended (see Figure 19 middle and left) by the following

Result: *S is the in-centre of triangle PQT. The circumcircle of PQT passes through the midpoints of a and b and of the circumcircle of triangle PQS.*

Another curiosity concerning circles is

Ebisui's six circle theorem (Figure 20): Let H be a star-shaped hexagon inscribed to a circle. Construct the six circles passing to two outer vertices and the inner corner between them. Then the three connections of the centres of opposite circles intersect in one point.

A simple analytic proof of this statement is due to B. Odehnal. It turns out that the intersection points of neighbouring "Ebisui circles" are on a conic. This property seems to be somehow related to the famous "Theorem of Miquel" (c.f. [7]), but is is still not clear, where Ebisui's discovery really belongs to.

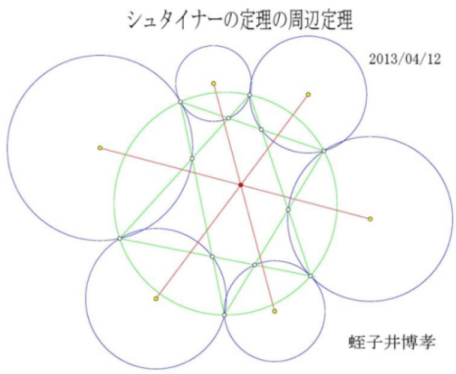

FIGURE 20. Original figure to Ebisui's six circle theorem

10. Elementary Geometry with circles: the Theorem of Miquel

The original version of the Theorem of Miquel concerns a triangle $\Delta = ABC$. Choose a point on each side of Δ, see Figure 21 (left, the points are labelled as X, Y, Z), then the circumcircles of the triangles AZY, BXZ, CYX pass through a common point, the "Miquel point" M.

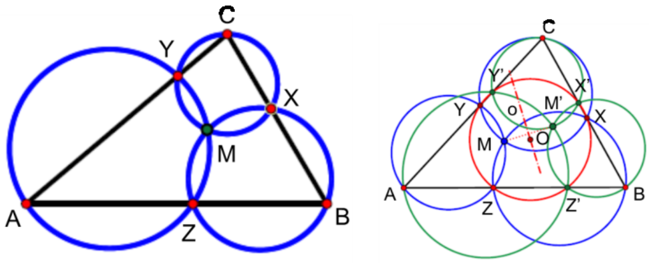

FIGURE 21. Elementary geometric Miquel figure (left) and Miquel mapping (right)

A first extension might consider also the circumcircle of XYZ. It intersects the sides of Δ in another triplet of points, see Figure 21 (right, the points are labelled as X', Y', Z') , which give rise to another Miquel point M'.

Result: *By the construction Figure 21 (right) a well-defined "Miquel mapping" $\mu : M \mapsto M'$ is established.*

Remark: The Miquel mapping μ is indeed a point to point mapping, that will say, to a fixed chosen M the point M' is also fixed for all possibly chosen points X, Y, Z. If Z moves on side AB, the centre O of the circumcircle of XYZ runs through a line o. It turns out that M' is the reflection image of M at o, see Figure 21 (right).

It is well known that Miquel's theorem is an axiom in circle geometries, c.f. e.g. [7] and [8]. To connect it to the elementary geometric version one has to interpret the Euclidean plane as Euclidean Mbius plane, that will say the sides of triangle Δ are Mbius circles, too, passing to the single ideal point U. Then Miquel's theorem reads as follows:

Theorem of Miquel: *Let a closed chain of four intersecting circles be given. If four of the eight intersection points are concyclic, then also the other four points are concyclic.*

It is interesting that the circle geometric version of Miquel's theorem is true also in e.g. the hyperbolic plane, while the elementary geometric version is false there, see Figure 22.

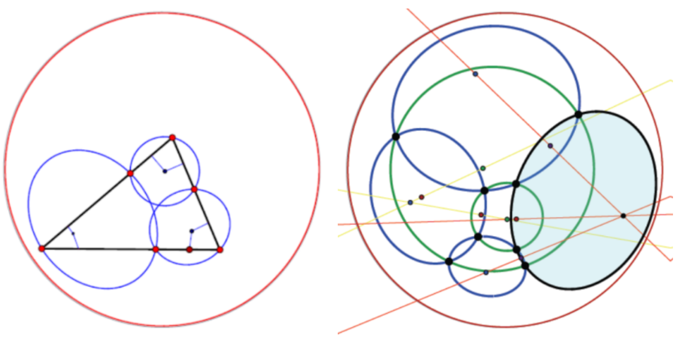

FIGURE 22. Visualisation of Miquel's theorem in the hyperbolic plane, left: elementary geometric version (false), right: circle geometric version (true)

11. Conclusion

This article aims at directing the attention of the reader to look again at often very well-known geometric material and use it as starting point for posing questions, which is the first and most important step of scientific creativity. The reader will also recognize the applied strategies for generalizations of mostly basic statements. Of course the sample of presented problems needs further extension, but this must be left to the - hopefully - curious reader. Elementary Geometry is an incredibly rich field and a never ending story and the list of references might give lots of hints for further research. Maybe Maths teachers should again use Elementary Geometry (in combination with a dynamic geometry software tool) to let their pupils discover the beauty of Geometry and Mathematics and learn them to be amazed of the often unexpected curiosities, which they might discover by themselves.

Finally it should be mentioned that all the figures in this paper are made by the "Cinderella 2.8 Geometry Software", which provides also standard models of (planar) Hyperbolic and Elliptic Geometry.

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