BRAHMAGUPTA'S THEOREM AUTOMATIC COMPUTER PROOF

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Abstract. The paper deals with a verification of Brahmagupta's theorem using dynamic geometry system and also with a proof of the theorem by classical way. Main part of the paper deals with a proof of the theorem by method of automatic theorem proving.

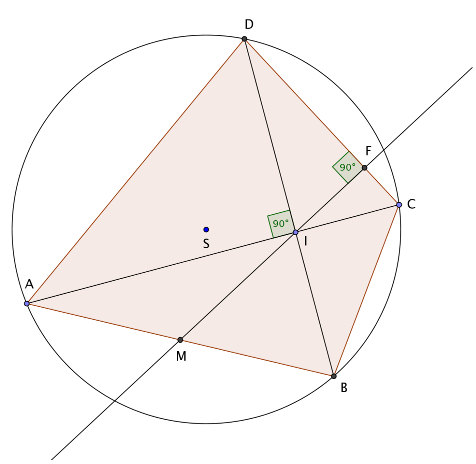
Key Words: Brahmagupta's theorem, automatic theorem proving.

1 Introduction

In this paper we will demonstrate how to prove Brahmagupta's theorem by computer. First we will describe and verify the theorem in dynamic geometry system GeoGebra. Further we will show a classical proof of this theorem. Finally we will prove this theorem by method of automatic theorem proving [1]. We will use a program CoCoA for this part.

2 Description of a problem

Theorem. *In a cyclic quadrilateral having perpendicular diagonals, a perpendicular to a side from the point of intersection of diagonals always bisects the opposite side.*



In a figure above we have a cyclic quadrilateral $ABCD$ that has perpendicular diagonals AC and BD. A perpendicular to a side CD goes through the intersection point I of diagonals and has a feet F.

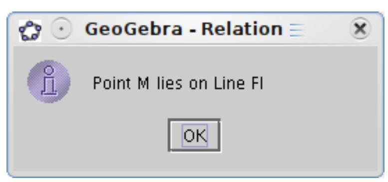
It is good to take a note here that in all another parts of this paper we will prove the theorem only for one side of cyclic quadrilateral. The proof for another sides of cyclic quadrilateral is analogical.

3 Verification in GeoGebra

First we have to construct the figure in GeoGebra. The steps of construction are as follows:

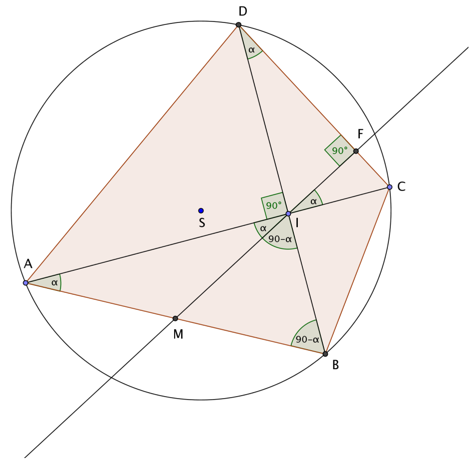
$$
\begin{aligned}
&1)k; k = (S, r)\\
&2)AC; A, C \in k\\
&3)BD; B, D \in k \wedge BD \perp AC\\
&4)I; I \in AC \cap BD\\
&5)FI; F \in CD \wedge FI \perp CD\\
&6)M; M \in AB \wedge |AM| = |MB|
\end{aligned}
$$

In this part we are to show that the midpoint M of side AB of the cyclic quadrilateral $ABCD$ belongs to perpendicular FI. In fact that is very easy. The only thing we have to do is to ask GeoGebra to find relation between two objects. In our case that is relation between the midpoint M and perpendicular FI. GeoGebra will tell us that the *point M lies on the line FI*. Thus the verification is done.



4 Classical proof

We are to prove that point $M \in FI \cap AB$ is a midpoint of side AB of the cyclic quadrilateral. We can also say that M is a midpoint of side AB if and only if the line FI divides right-angled triangle $\triangle BIA$ onto two isosceles triangles.



From the figure above we can see that

$$|\measuredangle CAB| = |\measuredangle CDB|.$$

The equality comes from properties of cyclic quadrilateral exactly from the inscribed angle theorem.

Next we have to realize that perpendicular FI divides triangles $\triangle\ CFI$ and $\triangle\ DIF$ onto two similar triangles.

$$|\measuredangle FIC| = |\measuredangle FDI| = |\measuredangle CDB|$$

Further we can see that $\measuredangle FIC$ and $\measuredangle MIA$ are vertical angles that are equal in size.

$$|\measuredangle FIC| = |\measuredangle MIA| \tag{1}$$

As we have said above the triangle $\triangle\ BIA$ is right-angled. It implies

$$|\measuredangle ABI| = 90 - |\measuredangle IAB| = 90 - |\measuredangle CAB| \text{ and } |\measuredangle BIM| = 90 - |\measuredangle MIA|. \tag{2}$$

Now we can see that (1) and (2) implies

$$|\measuredangle ABI| = |\measuredangle BIM|.$$

From above we can see that $\triangle\ IAM$ and $\triangle\ BIM$ are isosceles with common arm MI. This implies

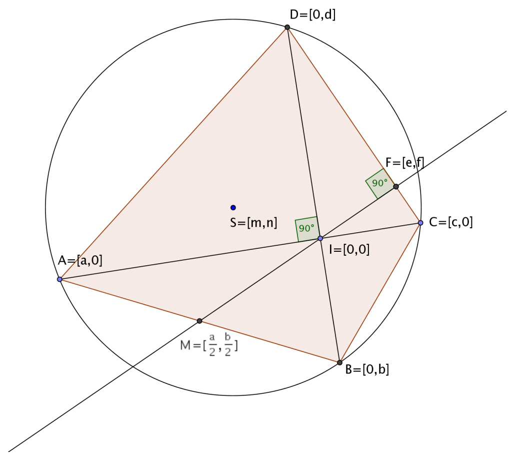
$$|AM| = |MI| = |MB|$$

so point M is a midpoint of side AB of cyclic quadrilateral $ABCD$. Hence the classical proof of Brahmagupta's theorem is done.

5 Automatic proof by computer

5.1 Introduction of a coordinate system

For this proof we will choose Cartesian coordinate system. As we can see from the figure below we denoted by $S = [m, n]$ the center of circle, by $A = [a, 0]$, $B = [0, b]$, $C = [c, 0]$, $D = [0, d]$ the vertices of cyclic quadrilateral, by $I = [0, 0]$ the point of intersection of diagonals AC and BD of cyclic quadrilateral, by $F = [e, f]$ the foot of perpendicular FI, and finally by $M = [\frac{a}{2}, \frac{b}{2}]$ the midpoint of side AB.



5.2 Algebraic formulation of a problem

First we have to translate geometric properties of objects into algebraic formulations. Like in classical proof we have some hypotheses $(h_1, \ldots, h_6)$ and a conclusion (c). We can express that four points belong to a circle with following equations. These equations come from Pythagorian theorem.

$$\begin{aligned}
r = |AS| \quad &\Leftrightarrow \quad h_1 : (a-m)^2 + n^2 - r^2 = 0 \\
r = |BS| \quad &\Leftrightarrow \quad h_2 : m^2 + (b-n)^2 - r^2 = 0 \\
r = |CS| \quad &\Leftrightarrow \quad h_3 : (c-m)^2 + n^2 - r^2 = 0 \\
r = |DS| \quad &\Leftrightarrow \quad h_4 : m^2 + (d-n)^2 - r^2 = 0
\end{aligned}$$

Foot F of perpendicular FI belongs to side CD of cyclic quadrilateral $ABCD$:

$$F \in CD \quad \Leftrightarrow \quad h_5 : de + cf - dc = 0$$

Side CD of cyclic quadrilateral $ABCD$ is perpendicular to line FI:

$$CD \perp FI \quad \Leftrightarrow \quad h_6 : ce - df = 0$$

We want to show that midpoint M of side AB belongs to line FI. We can express this relation by following equation:

$$M \in FI \quad \Leftrightarrow \quad c : fa - eb = 0$$

5.3 Proof of a statement

Now we will use computer algebra system CoCoA to do some hard work for us. First we have to tell CoCoA which indeterminates we will use so we enter:

```
Use R::=Q[a,b,c,d,e,f,m,n,r,t];
```

We want to find out whether conclusion polynomial c belongs to ideal generated by hypotheses polynomials $h_1, \ldots, h_6$. In CoCoA we enter:

```
I:=Ideal((a-m)^2+n^2-r^2, m^2+(b-n)^2-r^2, (c-m)^2+n^2-r^2,
m^2+(d-n)^2-r^2, de+cf-dc, ce-df);
NF(fa-eb, I);
```

We get

```
-be+af
```

as a result. If result is not equal to zero then it means that our conclusion polynomial does not belong to ideal I.

We will try a second method, the stronger criterion. We will ask CoCoA whether conclusion polynomial c belongs to a *radical* of ideal I. All we have to do is to add negation of conclusion to the set of generators of ideal I. Hence we get ideal J and we ask if 1 belongs to ideal J. In CoCoA we enter:

```
J:=Ideal((a-m)^2+n^2-r^2, m^2+(b-n)^2-r^2, (c-m)^2+n^2-r^2,
m^2+(d-n)^2-r^2, de+cf-dc, ce-df, (fa-eb)t-1);
NF(1, J);
```

We get

```
1
```

as a result. As in previous case if we get anything other than 0 as a result then it means that our conclusion polynomial is not an element of ideal J. Hence the statement is not generally true. It is necessary to look for additional conditions now.

5.4 Searching for additional conditions

In this step we have to find conditions for which the theorem is not meaningless. These conditions are called non-degeneracy conditions. These conditions are in form of in-equations ($\neq$) and are expressed only by independent indeterminates. To eliminate all dependent indeterminates and a slack variable t in ideal J, in CoCoA we enter:

```
Elim(e..t, J);
```

We get

```
Ideal(1/4abc - 1/4bc^2 - 1/4acd + 1/4c^2d,
1/4abd - 1/4bcd - 1/4ad^2 + 1/4cd^2)
```

as a result. This ideal is called elimination ideal. It is generated by two polynomials in our case. Now we will factor the first polynomial using CoCoA. We enter:

```
Factor(1/4abc - 1/4bc^2 - 1/4acd + 1/4c^2d);
```

We get

```
[[c, 1], [b - d, 1], [a - c, 1], [1/4, 1]]
```

as a result. These could be the desired conditions. What do they mean?

$c \neq 0$	...	If $c = 0$ then vertices B, C, D would be collinear.
$b - d \neq 0$	...	If $b = d$ then vertices B and D would coincide.
$a - c \neq 0$	...	If $a = c$ then vertices A and C would coincide.

Finally we have to add these conditions to ideal I. Thus we obtain ideal K and we can find a normal form of our conclusion polynomial c. In CoCoA we enter:

```
K:=Ideal((a-m)^2+n^2-r^2, m^2+(b-n)^2-r^2, (c-m)^2+n^2-r^2,
m^2+(d-n)^2-r^2, de+cf-dc, ce-df, c(b-d)(a-c)t-1);
NF(fa-eb, K);
```

We get

```
0
```

as a result. It means that conclusion polynomial c belongs to ideal K. Hence the theorem is generically true and the computer proof is done.

References

[1] Pech, P.: *Selected topics in geometry with classical vs. computer proving*. World Scientific Publishing, New Jersey London Singapore, (2007).