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MULTISTRUCTURES CREATED BY NON-INVARIANT SUBGROUPS OF THE GROUP OF LINEAR ORDINARY SECOND-ORDER DIFFERENTIAL OPERATORS

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Abstract. Following the study of application of algebraic methods in the theory of ordinary differential equations and their transformations we now construct and analyze multistructures of linear ordinary differential operators with continuous coefficients. Presented considerations form a natural motivation for the study of structures with one hyperoperation.

Introduction

Hypergroups and in particular join spaces play an important role in theories of various mathematical structures and their applications. The concept of a join space was introduced by Walter Prenowitz and used by him and James Jantosciak to reconstruct several branches of geometry – [4, 5, 7, 13, 14]. The other fields of applications of join spaces are lattices, graphs, ordered sets and automata. Noncommutative join spaces form an interesting subclass of the class of transposition hypergroups which satisfies a postulated property of transposition. For details cf. e.g. [13, 14]. More precisely, if H is a set, $\mathcal{P}(H)$ is the family of all subsets of H then a mapping $*\colon H \times H \to \mathcal{P}(H)$ is called a hyperoperation or join operation in H and the pair $(H, *)$ is said to be a hypergroupoid. The join operation is extended to subsets of H in a natural way, so that for $\emptyset \neq A \subset H$, $\emptyset \neq B \subset H$ the hyperproduct $A * B$ is given by $A * B = \bigcup\{a * b; a \in A, b \in B\}$. The relational notation $A \approx B$ (read A meets B) is used to assert that the sets A and B have nonempty intersection.

In H two hypercompositions – right extension "/" and left extension "\" – each being an inverse to $*$ are defined by $a/b = \{x; a \in x * b\}$ and $b\backslash a = \{x, a \in b * x\}$. Hence $x \approx a/b$ if and only if $a \approx x * b$ and $x \approx b\backslash a$ if and only if $a \approx b * x$.

The basic properties which the hypergroupoid might have include:

(1) $a * (b * c) = (a * b) * c$ for all $a, b, c \in H$ (Associativity),
(2) $a * H = H = H * a$ for all $a \in H$ (Reproduction),
(3) $b\backslash a \approx c/d$ implies $a * d \approx b * c$ for all $a, b, c, d \in H$ (Transposition).

An associative hypergroupoid is called a *semihypergroup*, a semihypergroup in which the reproduction axiom holds is called a *hypergroup*, a hypergroup in which transposition axiom holds is called a *transposition hypergroup* (*join space* in a commutative case) and a hypergroupoid which satisfies the reproduction axiom is called a *quasi-hypergroup.*

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The effort to study ordinary differential equations and their transformations is rooted in the work of collaborators and successors of Otakar Borůvka (1899 – 1995). One of his successors, František Neuman wrote in [19]: "Algebraic, topological and geometrical tools together with the methods of the theory of dynamical systems and functional equations make possible to deal with problems concerning global properties of solutions by contrast to the previous local investigations and isolated results." The influence of these ideas is a certain motivating factor of our investigations.

1. Preliminaries

We consider linear ordinary differential operators of the form

$$L(p,q) = D^2 + p(x)D + q(x)\mathrm{Id},$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$, $p, q \in \mathbb{C}(J)$, and groups and hyperstructures of such operators.

We have related some of our findings to quasi-ordered / partially ordered (semi)groups. By a *quasi-ordered semigroup* we mean a triple $(G, \cdot, \leq)$, where $(G, \cdot)$ is a semigroup and binary relation $\leq$ is a quasi-ordering (i.e is reflexive and transitive) on the set G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold. By a *partially ordered (semi) group* we mean (as usual) a triple $(G, \cdot, \leq)$, where $(G, \cdot)$ is a (semi)group and $\leq$ is a reflexive, anti-symmetrical and transitive binary relation on G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is a principal and generated by $a \in G$.

For this, the "Ends lemma" is used. It has the form of the following Theorems:

Theorem 1.1. *([7], Theorem 1.3, p. 146) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}'(S)$ defined by*

$$a * b = [a \cdot b)_{\leq}$$

is associative. The semi-hypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

Theorem 1.2. *([7], Theorem 1.4, p. 147) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

- 1^0**:** *For any pair $a, b \in S$ there exists a pair $c, c' \in S$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$*
- 2^0**:** *The associated semi-hypergroup $(S, *)$ is a hypergroup.*

Remark 1.3. If $(S, \cdot, \leq)$ is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S, \cdot, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.

Remark 1.4. The wording of the above Theorems is the exact translation of theorems from [7]. The respective proofs, however, do not change in any way, if we regard *quasi-ordered* structures instead of *partially ordered* ones as the anti-symmetry of the relation $\leq$ is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi–ordered structures.

The following theorem extending the "Ends lemma" was proved by Račková in her Ph.D. thesis. The proof can be also found in [21]. Notice that if $(H, \cdot)$ is commutative, then $(H, *)$ is a join space.

Theorem 1.5. *([21], Theorem 4) Let $(H, \cdot, \leq)$ be a quasi-ordered group and $(H, *)$ be the associated hypergroupoid. Then $(H, *)$ is the transposition hypergroup.*

When talking about mappings, by an *inclusion homomorphism* we mean a mapping $f\colon (G, \cdot_G) \to (H, \cdot_H)$ such that $f(a \cdot_H b) \subset f(a) \cdot_G f(b)$ for all pairs $a, b \in G$. If equality holds instead of inclusion and the mapping f is bijective then f is an isomorphism and we write $(G, \cdot_G) \simeq (H, \cdot_H)$.

In [8, 10, 11, 12] we have discussed hyperstructures related to second-order homogeneous linear differential equations with continuous coefficients. If we suppose that $J \subseteq \mathbb{R}$ is an open interval and $p : J \to \mathbb{R}$, $q : J \to \mathbb{R}$ are continuous functions, i.e. $p, q \in \mathbb{C}(J)$, where $p(x) > 0$ for all $x \in J$ and if we define

$$L(p, q)(f) = f''(x) + p(x)f'(x) + q(x)f(x),$$

i.e. $L(p, q)(y) = 0$ is a second order homogeneous linear differential equation with continuous coefficients in the form

$$y'' + p(x)y' + q(x)y = 0,$$

then we already know that the following proposition holds:

Proposition 1.6. *Let $J \subset \mathbb{R}$ be an open interval, $\mathbb{LA}_2(J) = \{L(p, q); p, q \in \mathbb{C}(J), p(x) > 0, x \in J\}$. For any pair of differential operators $L(p_1, q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ define*

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1p_2, p_1q_2 + q_1)$$

and $L(p_1, q_1) \leq L(p_2, q_2)$ if $p_1(x) = p_2(x)$, $q_1(x) \leq q_2(x)$ for any $x \in I$. Then $(\mathbb{LA}_2(J), \cdot, \leq)$ is a noncommutative partially ordered group with the unit element $L(\chi_1, \chi_0)$.

Having set for arbitrary pair of operators $L(p_1, q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ we put

$$\begin{aligned} L(p_1q_1) * L(p_2, q_2) &= \{L(p, q) \in \mathbb{LA}_2(J); L(p_1, q_1) \cdot L(p_2, q_2) \leq L(p, q)\} \\ &= \{L(p, q) \in \mathbb{LA}_2(J); L(p_1p_2, p_1q_2 + q_1) \leq L(p, q)\} \\ &= \{L(p_1p_2, q); \in \mathbb{C}(J), p_1q_2 + q_1 \leq q\}. \end{aligned}$$

and having applied the "Ends lemma" there has been obtained that

Theorem 1.7. *Let $J \subset \mathbb{R}$ be an open interval,$\mathbb{LA}_2(J) = \{L(p, q); [p, q] \in \mathbb{C}_+(J) \times \mathbb{C}(J)\}$ be the set of ordinary linear differential operators of second order — i.e. $L(p, q)(y) = y'' + p(x)y' + q(x)y = 0$, $y \in \mathbb{C}^2(J)$. If $L(p_1, q_1) * L(p_2, q_2) = \{L(p, q) \in \mathbb{LA}_2(J); p_1p_2 = p, p_1q_2 + q_1 \leq q\}$ for any pair $L(p_1q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ then $(\mathbb{LA}_2(J), *)$ is noncommutative transposition hypergroup, i.e. a noncommutative join space.*

Furthermore, in [1] there is given a certain construction of a quasi-hypergroup on systems of right and left cosets created by decompositions of groups of linear functions or affine transformations of linear space using the non-invariant subgroup isomorphic to the multiplicative group of non-negative real numbers or using the subgroup of linear transformations, i.e. endomorphisms of a linear space.

By Λ, R we denote the left / right equivalence on $\mathbb{LA}_2(J)$ determined by the non-invariant subgroup

$$\mathbb{L}_0\mathbb{A}_2(J) = \{L(p, 0); p \in \mathbb{C}(J), p(x) \neq 0, x \in J\}$$