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NORMAL SUBHYPERGROUPS OF HYPERGROUPS OF ORDINARY LINEAR SECOND-ORDER DIFFERENTIAL OPERATORS

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Abstract. Following the theory of hyperideals of hyperring there are investigated certain significant normal suphypergrups of hypergrups of second-order linear ordinary differential operators. In particulate there are treated basic algebraic properties of hypergrups of special differential operators in the Jacobi form.

In the theory of hyperrings there are investigated and used series of hyperideals [11]. In using composition series of hyperideals of hyperring there is generalised Jordan–Hölder theorem asserting that if a hyperring R has a composition series, then any two composition series for R are equivalent. So, a similarly important them is investigation of series of normal subhypergroups in hypergroups. In this contribution we concern ourselves onto certain significant normal subhypergrups of the hypergroup of ordinary linear differential operators-[1]. As usually, $\mathbb{C}^k(I)$ stands for the commutative ring of all real functions of one variable defined on an open interval I of reals, and having there continuous derivatives up to order $k \geq 0$. Instead of $\mathbb{C}^0(I)$ we write only $\mathbb{C}(I)$; this symbol denotes the ring of all continuous function on I, $\mathbb{C}_+(I)$ is its subsemiring of all positive continuous function. Denote by $\mathbb{A}_2(I)$ the set of non-singular ordinary linear homogeneous differential equations of the second order

$$y'' + p(x)y' + q(x)y \;=\; 0,$$

such that $p \in \mathbb{C}_+(I), q \in \mathbb{C}(I)$. Let us denote by Id the identity operator and $D = d/dx$. Further, we denote by $[p, q]$ as usually an ordered pair of functions p, q. By $L(p, q)$ will be denoted the differential operator $L(p, q) = D^2 + p(x)D + q(x)Id$, with the use of that the equation $(P_2(y, x; I))$ has the form $L(p, q)(y) = 0$. We denote by

$$\mathbb{LA}_2(I) \;=\; \{L(p, q) : \mathbb{C}^2(I) \longrightarrow \mathbb{C}(I); [p, q] \in \mathbb{C}_+(I) \times \mathbb{C}(I)\}$$

the set of all such differential operators.

For $r \in \mathbb{R}$ we denote by $\chi_r : I \longrightarrow \mathbb{R}$ the constant function with the value r.

Basic concepts of the mentioned theory can by found in [5, 9, 10, 13]. We remind that a hypergroup is pair $(H, \bullet)$, where $H \neq \emptyset$ and: $\bullet : H \times H \longrightarrow \mathcal{P}^\star(H)$ (the system of all non-empty subsets of H) is binary hyperoperation on H satisfying the associativity axiom: $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ for all $a, b, c \in H$ and the reproduction axiom: $a \bullet H = H = H \bullet a$ for any element $a \in H$. Here for any pair of non-empty subsets $A, B \subseteq H$ we define its hyperprotuct as $A \bullet B = \bigcup\{a \bullet b; a \in A, b \in B\}$. A set endowed with a binary hyperoperaion merely is termed a hypergroupoid. A

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subhypergroupoid of hypergroupoid $(H, \bullet)$ is a pair $(S, \bullet)$ where $S \bullet S \subseteq S$ i.e. the set then it is said to be a subhypergroup of $(H, \bullet)$. If $(H, \bullet), (K, \bullet)$ are hypergroupoids then a mapping $f : H \longrightarrow K$ is called homomorphism of the hypergroupoid $(H, \bullet)$ into the hypergoupoid $(K, \bullet)$ whenever it satisfies $f(x \bullet z) \subseteq f(x) \bullet f(y)$ for all pairs of elements $x, y \in H$. If $H = K$ then homomorphisms of $(H, \bullet)$ into $(H, \bullet)$ are called endomorphisms of $(H, \bullet)$.

We describe first the crucial simple construction from [5] which has been used also in [8] and involves to obtain similar - in certain sense - analogical results to those presented in this contribution.

By an order group we mean (as usually) a triad $(G, \bullet, \leq)$, where $(G, \bullet)$ is a group and $\leq$ is reflexive, symmetrical and transitive binary relation on the set G such that for any triad $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq y \bullet z, z \bullet x \leq z \bullet y$. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is a principal end generated by $a \in G$. In a group mappings $\lambda_a : G \longrightarrow G,\ \rho_a : G \longrightarrow G$ define by $\lambda_a(x) = a.x,\ \rho_a(x) = x.a$, are called a left translation, a right translation, respectively, determined by $a \in G$. Notice, that a group with an ordering $(G, \bullet, \leq)$ is an ordered group if and only if all its left and right translations $\lambda_a,\ \rho_a,\ a \in G$, are order-preserving i.e. isotone selfmaps of the ordered set $(G, \leq)$.

The following proposition is proved in [5]

Lemma 1. *Let $(G, \bullet, \leq)$ be an ordered group. Define a hyperoperation $* : G \times G \longrightarrow \mathcal{P}^\star(G)$ by*

$$a * b \ = \ [a \cdot b)_{\leq} (= \{x \in G; a \cdot b \leq x\})$$

for all pairs of elements $a,\ b\ \in G$. Then $(G, *)$ is a hypergroup which is commutative if and only if the group $(G, \bullet)$ is commutative. The hypergroup $(G, *)$ defined above will be called determined by the ordered group $(G, \bullet, \leq)$

Now we apply the simple construction of a hypergroup from Lemma 1 onto this considered concrete case of differential operators.

Proposition 2. *Let $I \subseteq \mathbb{R}$ be an open interval, $\mathbb{LA}_2(I) = \{L(p, q) : p, q \in \mathbb{C}(I), p(x) > 0, x \in I\}$. For any pair of differential operators $L(p_1, q_1), L(p_2, q_2) \in \mathbb{LA}_2(I)$ define*

$$L(p_1, q_1) \bullet L(p_2, q_2) \ = \ L(p_1 p_2, p_1 q_2 + q_1)$$

and $L(p_1, q_1) \leq L(p_2, q_2)$ if $p_1(x) = p_2(x), q_1(x) \leq g_2(x)$ for any $x \in I$. Then $(\mathbb{LA}_2, \bullet, \leq)$ is a non-commutative ordered group with the unit element $L(\chi_1, \chi_0)$.

Proof is contained in [5].

Proposition 3. *Let $I \subseteq \mathbb{R}$ be an open interval and let $\bullet : \mathbb{LA}_2(I) \times \mathbb{LA}_2(I) \longrightarrow \mathcal{P}^\star(\mathbb{LA}_2(I))$ be the above defined binary hyperoperation. Then the hypergroupoid $(\mathbb{LA}_2(I), \bullet)$ is a non-commutative hypergroup.*

A hypergroup $(H, \bullet)$ is called a transposition hypergroup or a join space if it satisfies the transposition axiom: For all $a, b, c, d, \in H$ the relation $b \setminus a \approx c/d$ implies $a.d \approx b.c$, (here $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$), where sets $b \setminus a = \{x \in H; a \in b.x\}, c/d = \{x \in H; c \in x.d\}$, are called left and right extensions or fraction, respectively.

We show that the above constructed hypergroup is transposition, thus that it is a non-commutative join space.

We need some auxiliary calculation:

Lemma 4. *Let $I \subseteq \mathbb{R}$ be an open interval, $L(f,u), L(g,v) \in \mathbb{LA}_2(I)$ be arbitrary operators, i.e. elements of the hypergroup $(\mathbb{LA}_2(I), \bullet)$. Then*

1) $L(f,u) \bullet L(g,v) \; = \; \{L(fg,q); q \in \mathbb{C}(I), f(x)v(x) + u(x) \leq q(x), x \in I\}$.
2) $L(f,u)/L(g,v) \; = \; \{L(\frac{f}{g},q); q \in \mathbb{C}(I), q(x) \leq \frac{1}{g(x)}(u(x)g(x) - f(x)v(x)), x \in I\} \subseteq \mathbb{LA}_2(I)$.
3) $L(g,v) \setminus L(f,u) \; = \; \{L(\frac{f}{g},q); q \in \mathbb{C}(I), q(x) \leq \frac{u(x)-v(x)}{g(x)}, x \in I\} \subseteq \mathbb{LA}_2(I)$.

Using of the above lemma we obtain the following theorem, which is proved in [5].

Theorem 5. *Let $I \subseteq \mathbb{R}$ by open interval, $\mathbb{LA}_2(I) \; = \; \{L(p,q); [p,q] \in \mathbb{C}_+(I) \times \mathbb{C}(I)\}$ be set of ordinary linear differential operators of second order - i.e. $L(p,q)(y) \; = \; y'' + p(x)y' + q(x)y = 0, y \in \mathbb{C}^2(I)$. If $L(p_1,q_2) * L(p_2,q_2) \; = \; \{L(p,q) \in \mathbb{LA}_2(I); p_1p_2 = p, p_1q_2 + q_1 \leq q\}$ for any pair $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ then $\mathbb{LA}_2(I),*)$ is non-commutative transposition hypergroup, i.e. a non-commutative join space.*

Translations in a group do not form endomorphisms of the considered group, however translation in $(\mathbb{LA}_2(I), \bullet)$ created by operators of the form $L(\chi_1, q)$ with $q \in \mathbb{C}(I), q \leq 0$ induce endomorphisms of the hypergroup $(\mathbb{LA}_2(I), *)$. Indeed, denoting Λ_q, R_q the left, right translation, respectively, determined by (χ_1, q) (here $\Lambda_q(L(u,v)) \; = \; L(\chi_1,q).L(u,v) \; = \; L(u, v+q)$ and dually for R_q) then since $L(\chi_1,q) \; \leq \; L(\chi_1,\chi_0)$ for any function $q \in \mathbb{C}(I), q \leq 0$ we obtain according to Lemma 2

Proposition 6. *Let $I \subseteq \mathbb{R}$ be open interval, $q \in \mathbb{C}(I), q(x) \leq 0$ for any $x \in I$. Then translations $\Lambda_q : \mathbb{LA}_2(I) \longrightarrow \mathbb{LA}_2(I), R_q : \mathbb{LA}_2(I) \longrightarrow \mathbb{LA}_2(I)$ determined by the operator $L(\chi_1, q)$ are bijective endomorphisms of the hypergroup $(\mathbb{LA}_2(I), *)$.*

Notice, that this Proposition 3 yields example of bijective endomorphisms of a hypergroup which are not automorphisms of the corresponding hypergroup. Further, with respect to Proposition 3 there is worthwhile to investigate other subset determined by specification of coefficients of consider second-order linear homogeneous differential equations. Thus we obtain some significant subsets of the set $\mathbb{LA}_2(I)$.

Let Denote

$$\mathbb{L}_{C1}\mathbb{A}_2(I) \; = \; \{L(c,q); L(c,q) \in \mathbb{LA}_2(I), c \in \mathbb{R}, c > 0\}$$

and

$$\mathbb{L}_{11}\mathbb{A}_2(I) \; = \; \{L(\chi_1,q); q \in \mathbb{C}(I)\}$$

We denote by "$*$" both restrictions of the corresponding hyperoperation for $\mathbb{LA}_2(I)$ onto the above defined subsets $\mathbb{L}_{C1}\mathbb{A}_2(I), \mathbb{L}_{11}\mathbb{A}_2(I)$. There is no danger of confusion. We are going to prove the below formulated theorem which can be consider as a certain generalization of Theorem 3 [1]. So the subset $\mathbb{L}_{11}\mathbb{A}_2(I)$ of $\mathbb{LA}_2(I)$ formed by all differential operators of the form $L(\chi_1,q), q \in \mathbb{C}(I)$ and $\mathbb{L}_{C1}\mathbb{A}_2(I)$ is formed by operators $L(p,q)$ with p - a constant function with positive value. Since $L(\chi_1,q_1) * L(\chi_1,q_2) \subseteq \mathbb{L}_{11}\mathbb{A}_2(I)$ for any pair of functions $q_1, q_2 \in \mathbb{C}(I)$ and with the hyperoperation "$*$" restricted on this subset, we see that$(\mathbb{L}_{11}\mathbb{A}_2(I), *)$ is commutative subhypergroupoid of the hypergroup $(\mathbb{LA}_2(I), *)$. Moreover we show that $(\mathbb{L}_{11}\mathbb{A}_2(I), *)$ is subhypergroup of the join space $(\mathbb{LA}_2(I), *)$ and it possess all of the following significant properties.

Definition 7. ([9],p.80). A subhypergroup $(S, .)$ of a hypergroup $(H, .)$ is called

- closed if $a/b \subseteq S$ and $b \setminus a \subseteq S$ for all $a, b \in S$,
- reflexive if $a \setminus S = S/a$ for all $a \in H$,
- normal if $a \cdot S = S \cdot a$ for all $a \in H$.

Remark 8. If a hypergroup S is a normal suphypergroup H we write $S \lhd H$.

In what follows we write $c, 1$ instead of χ_c, χ_1, respectively.

Theorem 9. *Let $I \subseteq \mathbb{R}$ be an open interval. The hypergroupoids $(\mathbb{L}_{11}\mathbb{A}_2(I), *), \mathbb{L}_{C1}\mathbb{A}_2(I), *)$ are suphypergroups of the hypergroup $\mathbb{L}\mathbb{A}_2(I)$ and we have*

$$\mathbb{L}_{11}\mathbb{A}_2(I) \lhd \mathbb{L}_{C1}\mathbb{A}_2(I) \lhd \mathbb{L}\mathbb{A}_2(I)$$

and

$$\mathbb{L}_{11}\mathbb{A}_2(I) \lhd \mathbb{L}\mathbb{A}_2(I).$$

Proof. Suppose $L(c_1, q_1), L(c_2, q_2) \in \mathbb{L}_{C1}\mathbb{A}_2(I)$.
Then

$$L(c_1, q_1) * L(c_2, q_2) = \{L(c_1c_2, g); q_1(x) + c_1q_2(x) \leq g(x), x \in I\} \in \mathbb{L}_{C1}\mathbb{A}_2(I).$$

Thus

$$\mathbb{L}_{C1}\mathbb{A}_2(I)*\mathbb{L}_{C1}\mathbb{A}_2(I) = \bigcup\{L(r,q)*L(s,v); L(r,q), L(s,v) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\} \subseteq \mathbb{L}_{C1}\mathbb{A}_2(I).$$

Evidently the hyperoperation "$*$" on $\mathbb{L}_{C1}\mathbb{A}_2(I)$ is associative and we show that the semihypergroup $(\mathbb{L}_{C1}\mathbb{A}_2(I), *)$ satisfies the reproduction axiom.

Suppose $L(c_1, p_1) \in \mathbb{L}_{C1}\mathbb{A}_2(I)$ is an arbitrary operator. Then evidently

$$L(c, p) * \mathbb{L}_{C1}\mathbb{A}_2(I) \subseteq \mathbb{L}_{C1}\mathbb{A}_2(I).$$

If $L(r, q) \in \mathbb{L}_{C1}\mathbb{A}_2(I)$ then define $s = \frac{r}{c}$ and $\varphi(x) = \frac{1}{c}(q(x) - p(x)), x \in I$. Then $\varphi \in \mathbb{C}(I)$ and

$$L(c, p) * L(s, \varphi) = \{L(cs, \psi); c\varphi(x) + p(x) \leq \psi(x), x \in I\} = \{L(r, \psi), c\varphi + p \leq \psi\}.$$

Since $q(x) = c\varphi(x) + p(x)$, we have

$$L(r, q) \in L(c, p) * L(s, \varphi) \subseteq L(c, p) * \mathbb{L}_{C1}\mathbb{A}_2(I),$$

thus

$$L(c, p) * \mathbb{L}_{C1}\mathbb{A}_2(I) = \mathbb{L}_{C1}\mathbb{A}_2(I).$$

Similarly, defining $\varphi(x) = q(x) - sp(x), x \in (I)$ we obtain $L(r, q) \in L(\frac{r}{c}, \varphi) * L(c, p) \subseteq \mathbb{L}_{C1}\mathbb{A}_2(I) * L(c, p)$, consequently

$$\mathbb{L}_{C1}\mathbb{A}_2(I) * L(c, p) = \mathbb{L}_{C1}\mathbb{A}_2(I).$$

Hence $(\mathbb{L}_{C1}\mathbb{A}_2(I), *)$ is a subhypergroup of the hypergroup $(\mathbb{L}\mathbb{A}_2(I), *)$. In the similar way we obtain quite analogical assertion for the hypergroupoid $(\mathbb{L}_{11}\mathbb{A}_2(I), *)$.

Now we show that for an arbitrary operator $L(p, q) \in \mathbb{L}\mathbb{A}_2(I)$ the equality

$$L(p, q) * \mathbb{L}_{C1}\mathbb{A}_2(I) = \mathbb{L}_{C1}\mathbb{A}_2(I) * L(p, q)$$

holds.

Suppose $L(p, q) \in \mathbb{L}\mathbb{A}_2(I)$ is an arbitrary operator.
We have

$$L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I) = \bigcup\{L(p,q) * L(c,v); L(c,v) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\} = \{L(f,g); L(cp, pv+q) \leq L(f,g); L(f,g) \in \mathbb{L}\mathbb{A}_2(I)\} = \{L(f,g); F(x) = c \cdot p(x), p(x)v(x) + q(x) \leq g(x), g \in \mathbb{C}(I)\}.$$

On the other hand

$$\mathbb{L}_{C1}\mathbb{A}_2(I) * L(p,q) \;=\; \bigcup\{L(r,u) * L(p,q); L(r,u) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\} \;=$$
$$\{L(\varphi,\psi); rp(x) = \varphi(x), rq(x) + u(x) \le \psi(x), \psi \in (I)\}.$$

Now, if $L(f,g) \in L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I)$, then $f(x) = \varphi(x)$ for $c = r$. Further, for $r = 1$ and $u(x) = p(x)v(x) \in \mathbb{C}(I)$ we have $rq(x) + u(x) = p(x)v(x) + q(x), x \in I$ thus $L(f,g) = L(\varphi,\psi)$ for

$$\varphi(x) = p(x)v(x) + q(x) = q(x) + u(x), x \in I, \text{i.e. L(f, q)} \in \mathbb{L}_{\mathrm{C1}}\mathbb{A}_2(\mathrm{I}) * \mathrm{L(p, q)},$$

hence

$$L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I) \subseteq \mathbb{L}_{C1}\mathbb{A}_2(I) * L(p,q).$$

Similarly,

$$\mathbb{L}_{C1}\mathbb{A}_2(I)*L(p,q) \;=\; \bigcup\{L(r,u)*L(p,q); L(r,u) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\} \;=\; \{L(\varphi,\psi); \varphi = rp, rq+u \le \psi\}.$$

The set on the left hand side is of the form

$$\bigcup\{L(f,g); f = cp, p\alpha + q \le g, L(c,\alpha) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\}$$

(for $c = r \in \mathbb{R}^+$ we have $f = \varphi$).

Suppose $L(u,v) \in \mathbb{L}_{C1}\mathbb{A}_2(I) * L(p,q)$. Then for some $L(c,\psi)$ we have $L(u,v) \in L(c,\psi) * L(p,q) = \{L(cp,\xi); cq + \varphi \le \xi, \xi \in \mathbb{C}(I)\}$, i.e. $u = cp, cq + \psi \le v$ Since $p(x) \ne 0$ for any $x \in I$, the function $h(x) = \frac{1}{p(x)}[(c-1)q(x) + \psi(x)], x \in I$ is well–defined, $h \in \mathbb{C}(I)$ and we obtain

$$p(x)h(x) + q(x) = cq(x) + \psi(x) \le v(x), x \in I,$$

hence

$$L(u,v) \in \bigcup\{L(cp,g); ph + q \le g, L(c,h) \in \mathbb{L}_{C1}\mathbb{A}_2(I)\} = \{L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I)\},$$

therefore

$$\mathbb{L}_{C1}\mathbb{A}_2(I) * L(p,q) \subseteq L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I).$$

Consequently, the equality

$$\mathbb{L}_{C1}\mathbb{A}_2(I) * L(p,q) = L(p,q) * \mathbb{L}_{C1}\mathbb{A}_2(I)$$

holds, i.e. the hypergroup $(\mathbb{L}_{C1}\mathbb{A}_2(I), *)$ is a normal subhypergroup of the hypergroup $(\mathbb{L}\mathbb{A}_2(I), *)$.In a similar way we obtain that also

$$(\mathbb{L}_{11}\mathbb{A}_2(I), *) \lhd (\mathbb{L}\mathbb{A}_2(I), *) \text{ and } (\mathbb{L}_{C1}\mathbb{A}_2(I), *) \lhd (\mathbb{L}\mathbb{A}_2(I), *).$$

□

In what follows we will consider linear differential operators in the so called Jacobi form. It is to be noted that Otakar Borůvka was the first who started in the fifties the systematic study of global properties of these equations including the global equivalence of these equations. O. Borůvka investigated the second order linear differential equations in the Jacobi form, i.e. $y" + p(x)y = 0$, where $p \in \mathbb{C}(I)$. In [14], p.229 there is proved that if h is the phase and φ the dispersion of the above equation (cf, [14]) then functions h and φ satisfy the Abel functional equation $h(\varphi(x)) = h(x) + \pi sgnh'$. So, with respect to importance of equations in the Jacobi form we will investigated hypergroups of operators $L(0,p), p \in \mathbb{C}(I)$.

So we will consider linear differential operators of the so called Jacobi form, i.e. $L(0,q) \in \mathbb{LA}_2(I)$ which acts in this way

$$L(0,q)y \;=\; y\text{''} + q(x)y, q \in \mathbb{C}(I).$$

It is to be noted that O. Borůvka in the year 1967 has obtained a criterion of the global equations of Jacobi form

$$y\text{''} + q(x)y \;=\; 0, q \in \mathbb{C}(I).$$

In detail see [1], or (certain survey) in [14]. In [1] there is proved the following theorem.

Theorem 10. *Let $I \in \mathbb{R}$ be an open interval and*

$$\mathbb{LA}_2(I)_q = \{L(p,q); p, q \in \mathbb{C}(I), q(x) \neq 0, x \in I\}, \mathbb{JA}_2(I)_q = \{L(0,q) \in \mathbb{LA}_2(I)_q\}.$$

For any pair of operators $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)_q$ define

$$L(p_1,q_1) \bullet_B L(p_2,q_2) \;=\; L(p_1q_2 + p_2, q_1q_2).$$

Then $(\mathbb{LA}_2(I)_q, \bullet_B)$ is a non-commutative group, with the unit $L(0,1)$ assigning to any function $f \in \mathbb{C}^2(I)$ the function $f\text{''} + f$.

Similarly we can define the other binary operation "$\bullet_O$" on the set $\mathbb{JA}_2(I)_q$ by $L(p_1,q_1) \bullet_O L(p_2,q_2) \;=\; L(p_1 + p_2q_1, q_1q_2)$. Then as above we obtain that $(\mathbb{JA}_2(I)_q, \bullet_O)$ is a commutative subgroup of the group $(\mathbb{LA}_2(I)_q, \bullet_O)$. However does not hold $(\mathbb{JA}_2(I)_q, \bullet_B) \lhd (\mathbb{LA}_2(I)_q, \bullet_B)$ as well as $(\mathbb{JA}_2(I)_q, \bullet)$ is not normal subgroup of $(\mathbb{LA}_2(I)_q, \bullet_O)$. Indeed, for arbitrary operators $L(p,q) \in \mathbb{LA}_2(I)_q$ and $L(0,u) \in \mathbb{JA}_2(I)_q$ we have

$$L^{-1}(p,q) \bullet_B L(0,u) \bullet_B L(p,q) \;=\; L(-\tfrac{p}{q}, \tfrac{1}{q}) \bullet_B L(p, qu) \;=\; L(p - \tfrac{p}{q} \cdot qu, u) \;=\; L(p - pu, u) \notin \mathbb{JA}_2(I)_q \text{ if } u \neq 1.$$

Similarly for the pair $(\mathbb{JA}_2(I)_q, \bullet_O), (\mathbb{LA}_2(I)_q, \bullet_O)$ we obtain the other fact above. If we denote $\mathbb{J}_C\mathbb{A}_2(I)_q \;=\; \{L(0,r); r \in \mathbb{R}, r \neq 0\}$ then with respect to Theorem 2 [1] we have that subgroupoid $(\mathbb{J}_C\mathbb{A}_2(I)_q, \bullet_B)$ of the group $(\mathbb{JA}_2(I)_q, \bullet_B)$ is its normal commutative subgroup. Notice that operations "$\bullet_B$" "$\bullet_O$" on the set $\mathbb{J}_C\mathbb{A}_2(I)q$ coincide.

We construct hypergroupoids $(\mathbb{JA}_2(I)_q, *_B), (\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ by defining hyperoperations:

$$*_B : \mathbb{JA}_2(I)_q \times \mathbb{JA}_2(I)_q \longrightarrow \mathcal{P}(\mathbb{JA}_2(I)_q)$$
$$*_C : \mathbb{J}_C\mathbb{A}_2(I)_q \times \mathbb{J}_C\mathbb{A}_2(I)_q \longrightarrow \mathcal{P}(\mathbb{J}_C\mathbb{A}_2(I)_q)$$

in this way:

For arbitrary pair $L(0,p), L(0,q) \in \mathbb{JA}_2(I)_q$ we put

$$L(0,p) *_B L(0,q) \;=\; \{L(0,\varphi); \varphi \in \mathbb{C}(I), p(x).q(x) \leq \varphi(x), x \in I\}$$

and for arbitrary pair $L(0,r), L(0,s) \in \mathbb{J}_C\mathbb{A}_2(I)_q$ we define

$$L(0,r) *_C L(0,s) \;=\; \{L(0,t); t \in \mathbb{R}^+, r.s \leq t\}$$

Using the following lemma (proof of which is e.g. in [8]) we obtain that the groupoids $(\mathbb{JA}_2(I)_q, *_B), (\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ are hypergroups.

Lemma 11. *In a commutative hypergroupoid $(H, \cdot)$ the following assertion are equivalent.*

1) *There holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any triad $a, b, c \in H$.*
2) *There holds $(a \cdot b) \cdot c \subseteq a \cdot (b \cdot c)$ for any triad $a, b, c \in H$.*

3) *There holds* $a \cdot (b \cdot c) \subseteq (a \cdot b) \cdot c$ *for any triad* $a, b, c \in H$.

Proof is contained in [8].

Theorem 12. *Let* $I \subseteq \mathbb{R}$ *be an open interval. Hypergruopoids* $(\mathbb{J}\mathbb{A}_2(I)_q, *_B), (\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ *are commutative hypergroups such that* $(\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ *is normal subhypergroup* $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$.

Proof. Suppose $J \subseteq \mathbb{R}$. For arbitrary pair $L(0,p), L(0,q) \in \mathbb{J}\mathbb{A}_2(I)_q$ we have $L(0,p) *_B L(0,q) = \{L(0,\varphi); \varphi \in \mathbb{C}(I), p(x).q(x) \leq \varphi(x), x \in I\} = L(0,q) *_B L(0,p)$. Clearly, the hypergroupoid $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$ is commutative. Let $L(0,p), L(0,q), L(0,u) \in \mathbb{J}\mathbb{A}_2(I)_q$ be arbitrary triad. Suppose

$$L(0,\varphi) \in (L(0,p) *_B L(0,q)) *_B L(0,u) = \bigcup\{(L(0,\psi) *_B L(0,u); L(0,\psi) \in L(0,p) *_B L(0,q)\}.$$

Then there exists a function $\psi_0 \in \mathbb{C}(I), \psi_0(x) \geq 0, x \in I$ such that $p(x)q(x) \leq \psi_0(x), x \in I$ and $L(0,\varphi) \in L(0,\psi_0) *_B L(0,u)$ i.e. $\psi_0(x) \cdot u(x) \leq \varphi(x), x \in I$. Then $p(x) \cdot q(x) \cdot u(x) \leq \varphi, x \in I$. For $\omega_0(x) = q(x) \cdot u(x), x \in I$ we have

$$p(x) \cdot \omega_0(x) = p(x) \cdot q(x) \cdot u(x) \leq \varphi(x), x \in I$$

thus

$$L(0,\varphi) \in L(0,p) *_B L(0,\omega_0) \subseteq \bigcup\{L(0,p) *_B L(0,\omega_0); L(0,\omega_0) \in L(0,q) *_B L(0,u)\} = L(0,p) *_B (L(0,q) *_B L(0,u)).$$

Hence

$$L(0,\varphi) \in L(0,p) *_B (L(0,q) *_B L(0,u)).$$

Since inclusions

$$(L(0,p) *_B L(0,q)) *_B L(0,u) \subseteq L(0,p) *_B (L(0,q) * L(0,u))$$

for any triads $[L(0,p), L(0,q), L(0,u)] \in \mathbb{J}\mathbb{A}_2(I)_q \times \mathbb{J}\mathbb{A}_2(I)_q \times \mathbb{J}\mathbb{A}_2(I)_q$ imply the opposite inclusions, we have obtained that the hypergroupoid $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$ is associative i.e. it is a semihypergroup. For $\omega_0(x) = q(x) \cdot u(x), x \in I$ we have

$$p(x) \cdot \omega_0(x) = p(x) \cdot q(x) \cdot (x) \leq \varphi(x), x \in I$$

thus

$$L(0,\varphi) \in L(0,p) *_B L(0,\omega_0) \subseteq \bigcup\{L(0,p) *_B L(0,\omega); L(0,\omega) \in L(0,q) *_B L(0,u)\} = L(0,p) *_B (L(0,q) *_B L(0,u)).$$

Hence

$$L(0,\varphi) \in L(0,p) *_B (L(0,q) *_B L(0,u)).$$

Futher $L(0,p) *_B \mathbb{J}\mathbb{A}_2(I)_q = \mathbb{J}\mathbb{A}_2(I)_q$. Evidently $L(0,p) *_B \mathbb{J}\mathbb{A}_2(I)_q \subseteq \mathbb{J}\mathbb{A}_2(I)_q$. Consider an arbitrary operator $L(0,\varphi) \in \mathbb{J}\mathbb{A}_2(I)_q$. There exists

$$L(0,\psi) \in \mathbb{J}\mathbb{A}_2(I)_q : L(0,\varphi) \in L(0,p) *_B L(0,\psi), p \cdot \psi \leq \varphi$$

define a function $\psi(x) = \frac{\varphi(x)}{p(x)}, x \in I$. Since $p(x) > 0$ for any $x \in I$ the function $\psi : I \longrightarrow \mathbb{R}$ is well–defined continuous function and positive on the interval I. Further

$$p(x) \cdot \psi(x) = \frac{p(x)\varphi(x)}{p(x)} = \varphi(x), x \in I \text{ thus } L(0,\varphi) \in L(0,p) *_B L(0,\psi) \text{ where } L(0,\psi) \in \mathbb{J}\mathbb{A}_2(I)_q,$$

consequently

$$\mathbb{J}\mathbb{A}_2(I)_q \subseteq L(0,p) *_B \mathbb{J}\mathbb{A}_2(I)_q$$

Therefore the reproduction axiom is satisfied.

Now, consider the hypergrupoid $(\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$. Since for every pair of operators $L(0,r), Ł(0,s) \in \mathbb{J}_C\mathbb{A}_2(I)_q \subset \mathbb{J}\mathbb{A}_2(I)_q$ it holds

$$L(0,r) *_C L(0,s) = \{L(0,t); t \in \mathbb{R}^+, r \cdot s \leq t\} = L(0,r) *_B L(0,s) = \{L(0,\varphi); \varphi \in \mathbb{C}_+(I); r \cdot s \leq \varphi(x), x \in I\} \cap \mathbb{J}_C\mathbb{A}_2(I)_q$$

(hence $\varphi \in \mathbb{C}_+(I)$) are positive constant functions only which can be identify with their values) we have that $(\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ is a subhypergroupoid of the hypergroup $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$. The hyperopertion "$*_C$" is evidently associative, commutative. Similarly as above (using positive real numbers representing positive constant function only) we obtain the inclusion

$$\mathbb{J}_C\mathbb{A}_2(I)_q \subset L(0,r) *_C \mathbb{J}_C\mathbb{A}_2(I)_q$$

is valid for arbitrarily chosen operator $L(0,r) \in \mathbb{J}_C\mathbb{A}_2(I)_q$. Since the opposite inclusion is evident, we have that the semihypergroup $(\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ is a commutative subhypergroup of the hypergroup $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$. Since the hypergroup $(\mathbb{J}\mathbb{A}_2(I)_q, *_B)$ is commutative its subhypergroup $(\mathbb{J}_C\mathbb{A}_2(I)_q, *_C)$ is normal. The proof is complete. □

Considering the binary operation on the set $\mathbb{J}\mathbb{A}_2(I)$ defined by $L(0,p) \cdot L(0,q) = L(0, p \cdot q); p \neq 0, q \neq 0$ we have that this structure is a commutative group. Since the considered hypergroup $(\mathbb{J}\mathbb{A}_2(I), *_B)$ is obtained using the Ends lemma then in the case of the group $\mathbb{J}^+\mathbb{A}_2(I) = \{L(0,p); p > 0\}$ we obtain by [15]. Theorem 12 that $(\mathbb{J}^+\mathbb{A}_2(I), *_B)$ is a join space. In general, the question weather $(\mathbb{J}\mathbb{A}_2(I), *_B)$ is a join space seems to be open. However, according to the following theorem the hypergroup $(\mathbb{L}\mathbb{A}_2(I), *_C)$ is closed and reflexive. Recall that a subhypergrouup S of the hypergroup H is called closed (reflexive) if $a/b \subseteq S$ and $b \setminus a \subseteq S$ for all $a, b \in S$ ($a \setminus S = s/a$ for all $a \in H$).

Theorem 13. *Let $I \subseteq \mathbb{R}$ be open interval. The subhypergroup $(\mathbb{J}_C\mathbb{A}_2(I), *_C)$ of the hypergroup $(\mathbb{J}\mathbb{A}_2(I), *_B)$ is closed and reflexive.*

Proof. Since the Hypergroup $(\mathbb{J}\mathbb{A}_2(I), *_B)$ is commutative we have for all pairs of operators $L(0,r), L(0,s) \in \mathbb{J}_C\mathbb{A}_2(I)$ that

$$L(0,r)/L(0,s) = L(0,s) \setminus L(0,r).$$

Further

$$L(0,r)/L(0,s) = \{L(0,\alpha); L(0,r) \in L(0,\alpha) *_C L(0,s)\} = \{L(0,\alpha); \alpha \cdot s \leq r, \alpha \neq 0\} \subseteq \mathbb{J}_C\mathbb{A}_2(I).$$

Thus the suphypergroup $(\mathbb{J}_C\mathbb{A}_2(I), *_C)$ of the hypergroup $(\mathbb{J}\mathbb{A}_2(I), *_B)$ is closed.

Similarly, if $L(0,q) \in \mathbb{J}\mathbb{A}_2(I)$ then

$$L(0,q) \setminus \mathbb{J}_C\mathbb{A}_2(I) = \bigcup_{L(0,r) \in \mathbb{J}_C\mathbb{A}_2(I)} (L(0,r)/L(0,q)) = \mathbb{J}_C\mathbb{A}_2(I)/L(0,q)$$

there-fore the subhypergroup $(\mathbb{J}_C\mathbb{A}_2(I), *_C)$ is reflexive. □

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