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INVERSION AND PROBLEM OF TANGENT SPHERES

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Abstract. The locus of centers of circles tangent to two given circles in plane is known to be a pair of conic sections. The foci of these conic sections are the centers of the circles given. As a generalization we get the Apollonius task with one missing element. In a spatial generalization of the problem mentioned we are to find a locus of centers of spheres tangent to three given elements (spheres, planes, or incident points). This locus consists of the intersections of pairs of quadric surfaces of revolution, their foci being the centers of spheres given. These intersections are known to be composed of conics. Some special configurations of elements given result in a task clear and easy even for high-school students. Sphere inversion helps to find the loci of points of tangency of the spheres of a parametric system to be found with elements given, whilst the locus of centers must be constructed using some other methods.

Introduction

Differences between high and low achievers increase demands on teachers. They must cope with the following facts:

- Most students need slow pace in progress.
- Those, who have low spatial abilities, need detailed instruction and visual models.
- Some students need a manual experience with 3D objects first, for some it is difficult to accept higher level of abstraction.
- Best students tend to be bored.

Interactive, dynamic geometry systems are an inventive tool in geometry, they can make students deal with "new" problems and can check the correctness of students' answers immediately. They allow the teacher to enhance his geometry lessons by constructions of three-dimensional objects. Three-dimensional analogues of planar constructions can be a suitable sort of such tasks. Construction of a sphere tangent to three given surfaces is a new problem to most students and it compels them to look for new ideas. A 3D dynamic geometry system helps them to obtain (virtually) real scene with the searched object constructed without necessity to spend a lot of time drawing the scene desired.

1. Problem of tangent circle and its 3D extension

Let's start with a known planar problem: *Find a circle tangent to two given tangent circles and a given straight line tangent to both the circles given.* It is an easy special configuration of the Apollonius task "line, circle, circle" that can be solved e.g. using construction of an algebraic expression. Since the length of the

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common tangent segment to two tangent circles of radii r_1, r_2 is $|T_1T_2| = 2\sqrt{r_1 r_2}$ and $h = |T_1T_2|$, we can derive the formula for circles searched:

$$\rho_{1,2} = \frac{h^2}{4\left(\sqrt{r_1} \pm \sqrt{r_2}\right)^2}.$$

In GeoGebra we can calculate radii of the two resulting circles or construct these radii using the compasses and ruler tools as well (Figure 1).

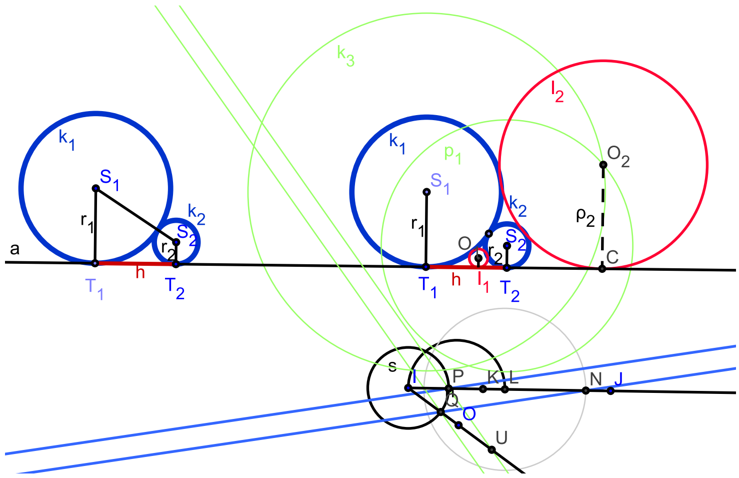

FIGURE 1. Construction of radii of the tangent circles.

The task mentioned above can be solved using an inversion in circle in its general configuration. It is a quite effective and nice method and it leads to the truly easy construction of a circle tangent to two parallel straight lines and a circle inscribed in our special configuration. Figure 2 was constructed in GeoGebra, too.

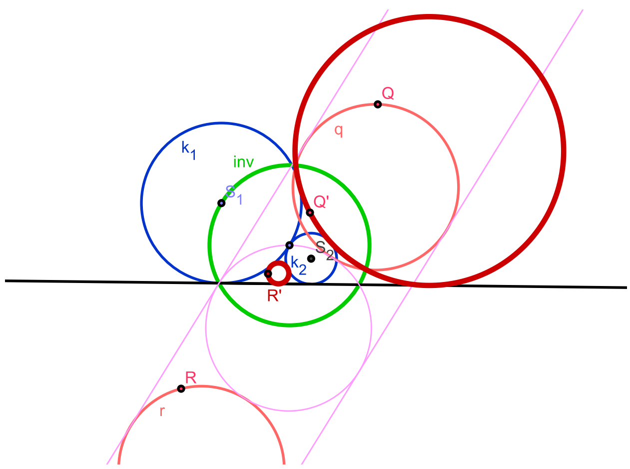

FIGURE 2. Task solved using an inversion.

Being extended to the three-dimensional space, the problem becomes more interesting. We are to look for all spheres tangent to two given spheres and one plane, all three surfaces given being tangent to each other. It is evident that we get a system of spheres. But it is difficult to guess how this system will look like without any previous experience with such problems. First, let's consider this problem in general.

2. General problem – set of tasks

Our problem can be considered as the Apollonius task with one given element missing.

2.1. System of circles tangent to two given elements. A planar problem is easy to discuss even for high-school students and we come to the following conclusion:

A locus of centers of circles tangent to two given elements (lines, circles, or passing through given points) is a straight line (two lines) or a conic section (two conics).

Take all the cases separately, there are six of them and the loci are:

- *point, point* – perpendicular bisector
- *point, straight line* – parabola
- *point, circle* – conic section
- *line, line* – a pair of straight lines
- *line, circle* – two parabolas
- *circle, circle* – a pair of confocal conics, their main axis (major or transverse) being $\frac{1}{2}(r_1 \pm r_2)$ (see Figure 3, [1]). Each conic section corresponds to one kind of tangency. If the circles given are tangent, the result is only one conic (and one straight line – centers of circles tangent to the circles given in their point of tangency).

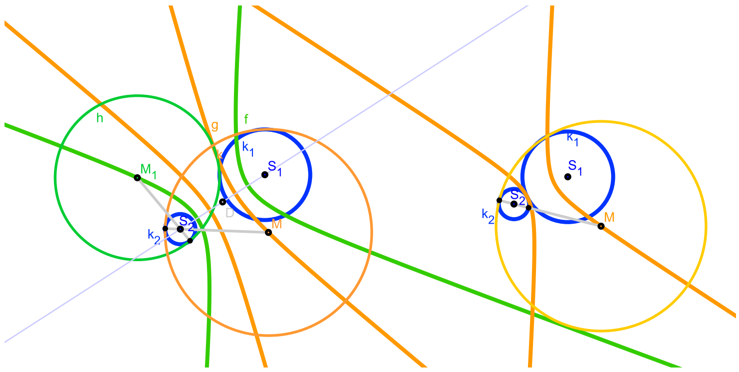

Figure 3. Loci of centers of spheres tangent to two given circles.

2.2. System of spheres tangent to three given surfaces. Three-dimensional analogues of the Apollonius task with one given element missing (i.e. spheres tangent to three given planes, spheres, or passing through given points). It is more difficult to see an answer. Is it similar to the planar problem? Take all the ten cases separately again and discuss the loci of centers.

- *point, point, point* – straight line
- *point, point, plane* – plane section of a paraboloid of revolution, thus a conic section
- *point, point, sphere* – plane section of a quadric surface of revolution, thus a conic section
- *point, plane, plane* – plane section of a paraboloid of revolution, thus conic sections (two conics)
- *point, plane, sphere* – intersection of quadric surfaces of revolution – a curve of degree four in general, a new problem
- *point, sphere, sphere* – intersection of quadric surfaces of revolution
- *plane, plane, plane* – 4 straight lines
- *plane, plane, sphere* – plane section of a paraboloid of revolution, thus a conic section (2 planes of symmetry, two paraboloids, four conics)
- *plane, sphere, sphere* – intersection of quadric surfaces of revolution
- *sphere, sphere, sphere* – intersection of quadric surfaces of revolution

Since the results in four of the cases discussed seem to be curves unknown to students, we must examine them closely. As mentioned above, the locus of centers of circles tangent to two circles are two confocal conics. As an analogy in 3D (conics rotating around the line passing through the centers of given circles/spheres), we have got a quadric surface of revolution.

Foci of all the quadrics are centers of the spheres given, thus pairs of the intersecting quadrics mentioned have a common focus. Such intersections are known to be composed of conics. [2]

We can thus conclude: *The loci of centers of all spheres tangent to three given elements (spheres, planes or incident points) are composed of (straight lines or) conics.*

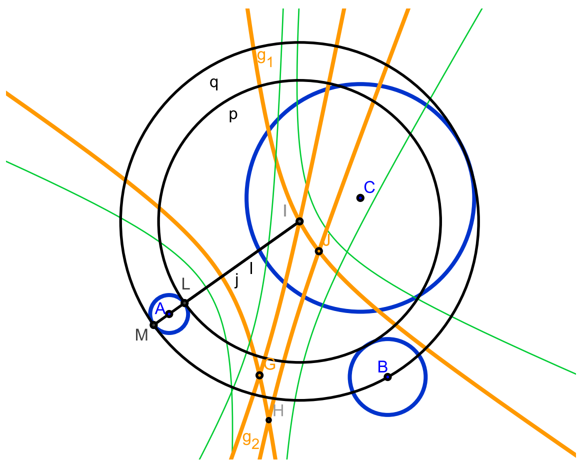

FIGURE 4. Only two of four points of intersection fulfill the task assignment. Point I does not, none of the circles p, q is tangent to all the three circles given.

It is important to stress the fact, that whilst an intersection of two such quadric surfaces is composed of two conics, the answer to the problem studied gives only one of them. Each of the quadrics is a locus of centers of spheres that have the same (both inner or both outer) or different (one inner and the other outer) kind of tangency to the spheres given. We can illustrate the problem in its planar analogues in Figure 4 where only two of four possible points of intersection of two conics (both in orange color) satisfy the desired property as centers of the circles tangent to all three circles given (in blue).

These loci are (in general) composed of up to four conics, their construction being quite time consuming and the final scene being confused. The special configuration, in which the elements given are mutually tangent, simplifies the task a lot, the conic section being only one (and one straight line).

3. Solving the particular problem in Cabri 3D

To avoid further inconveniences and special cases we will put a particular formulation of given problem:

Two externally tangent spheres of different radii and their common tangent plane are given. Find the locus Λ *of centers of all spheres tangent to the two given spheres and to the given plane. Find the loci* Λ_1, Λ_2, Λ_3 *of points of tangency on the given surfaces, too.*

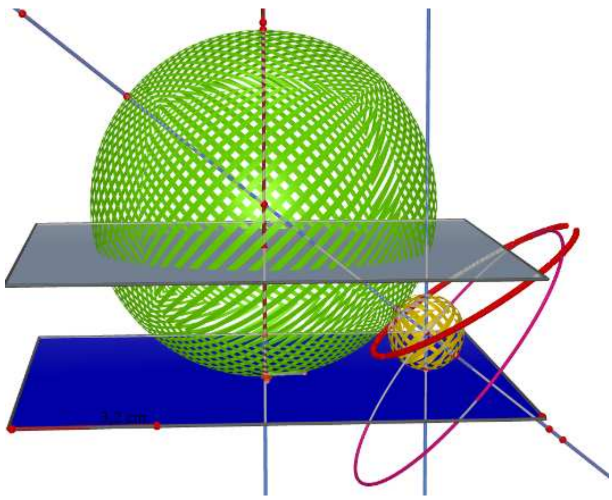

Figure 5. Guessing a locus of centers

3.1. **Answer not using inversion.** Since inversion (in circle/in sphere) is not a compulsory part of high-school geometry curriculum, we will try to answer all the questions using other ideas.

At the very beginning we may let students guess the loci searched with the help of dynamic geometry software. The construction of a center of such a sphere of given radius is easy, and changing the radius given dynamically, the Locus tool will give us the hint: The locus seems to be an ellipse as we can see in Figure 5.

By analogy with the planar problem mentioned above, we can reveal the fact that the locus of centers of the spheres tangent to a given plane and to a given sphere is a paraboloid of revolution (two paraboloids, in general). Its axis is perpendicular to the plane given and passes through the given sphere center. Thus the locus desired is a curve of intersection of two paraboloids of revolution with parallel axes. It is interesting to observe that this intersection is planar. We can show it using analytic geometry or by the following theorem of projection: *The elliptic section of a paraboloid of revolution is orthogonally projected into the plane orthogonal to the axis of the paraboloid into a circle.* We see that the plane (let it be α) of axes of the paraboloids is the plane of symmetry of this section.

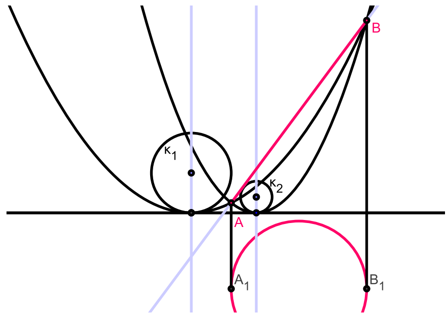

FIGURE 6. Intersection of two paraboloids of revolution

Plane sections of paraboloids by a plane perpendicular to α passing through the "lowest" and the "highest" crossing points A, B (if they exist) are ellipses with the major axis AB. It is evident that the projection of AB into the given plane is the diameter of both the circles to which these ellipses are projected (see Figure 6). Thus these circles coincide and the ellipses of the two planar sections coincide, too, being thus the intersection of the paraboloids.

Moreover, the projection of this ellipse into the given plane – circle k – is the locus Λ_1 of points of tangency of spheres with the plane given.

Finally, each point of tangency of a given sphere κ and the sphere ν found is their center of homothecy that transforms a given plane (tangent to κ) to a parallel plane tangent to ν in their point of tangency P. It is known that a stereographic projection preserves circles (not passing through P). Thus the loci of tangency on the spheres given are circles.

3.2. **Simplifying the problem inverting it in a sphere.** Inversion in a sphere, in which the main sphere center is the common point of spheres given, transforms this complicated task to quite a simple one. It inverts the spheres given into a pair of parallel planes tangent to a sphere – the image of the given plane. Now we easily construct a system of spheres tangent to the three surfaces inverted.

Since inversion preserves tangency, the image of this system of spheres in inversion is the system searched. We will construct loci Λ'_1, Λ'_2, Λ'_3 of points of tangency on the images of surfaces given. Their images are the loci searched on surfaces given. Construction of the inversion, the images of spheres given and the answer to the transformed task is in Figure 7. A construction of the center of a sphere searched is in the bottom part of this diagram.

It is important to stress the fact that the centers of spheres searched cannot be obtained using inversion. The locus of centers of spheres – an ellipse – has to be constructed by any proper planar construction using points on Λ_1, Λ_2, Λ_3.

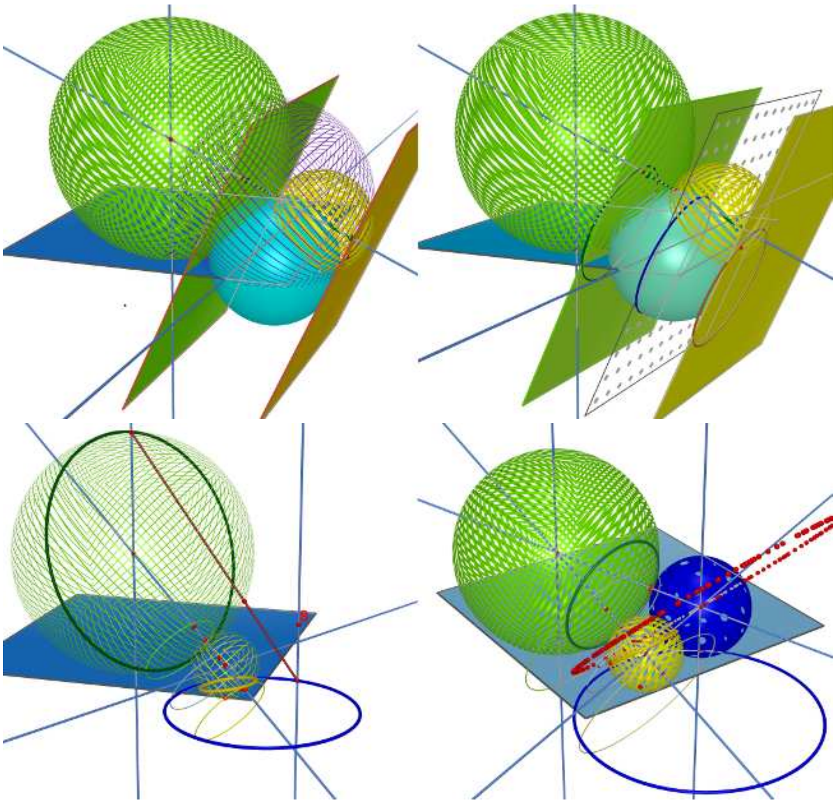

Figure 7. Inversion in a sphere and the answer to the task, locus Λ.

4. One more problem...

Inversion gives an answer to the seemingly more complicated task: Problem of four tangent spheres. We will change a plane in the previous task for a sphere tangent to both the spheres given. In fact, it is not a new problem at all, because it is possible to find an inversion that inverts one task to the other. But if we are not aware of this fact, a proper sphere inversion transforms the task to the previous inverse configuration. Constructing the main sphere of inversion orthogonal to one of the spheres given simplifies the scene, keeping this sphere fixed.

Though the problem seems to be cleared up, three important comments are to be said:

- The orthogonal sphere is a double (fixed) sphere in inversion but its circle of points of tangency with the spheres searched is not! (See Figure 8.)
- The result – conic section – is not the same as before (an ellipse). In our configuration it is a hyperbola (see Figure 8).
- Spheres with infinite radius – planes – are included in the resulting system of spheres.

Construction of all planes tangent to the three given spheres is a nice and easy task, too. It is interesting to solve it in general, with the given spheres in any configuration.

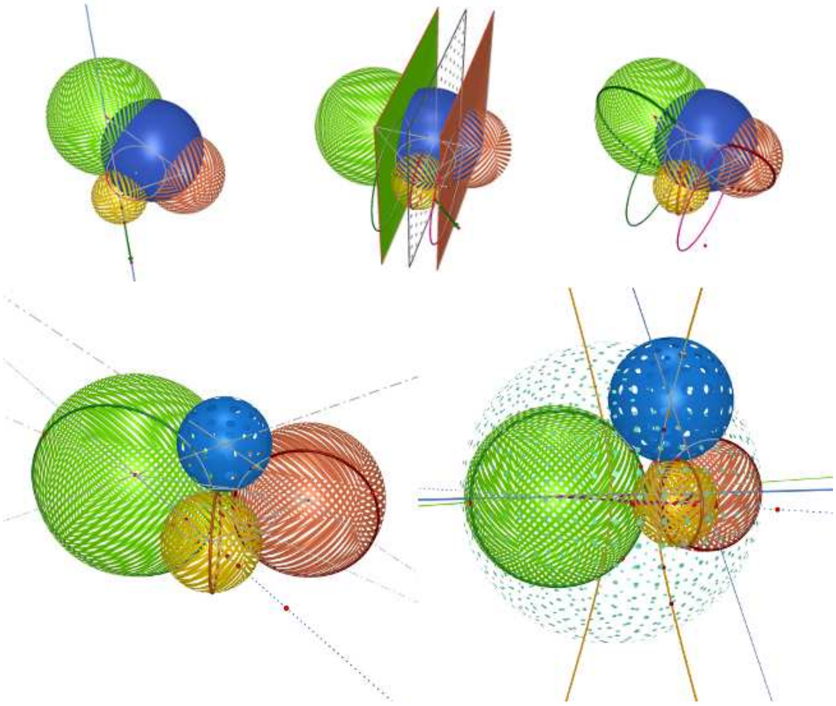

FIGURE 8. Problem of four tangent spheres.
Using an inversion in construction and examples of answer spheres.

CONCLUSION

Getting dynamic scene with the searched object constructed can be very motivating for students. An immediate graphic response of the system makes them satisfied and confident in the correctness of the result. Such a result can not be obtained via manual manipulation with a solid model.

A teacher has to prepare carefully the starting scene that can be dynamically modified later, though. An empty scene is confusing and building the scene is time-consuming for the students.

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REFERENCES

[1] J. Holubář: *O methodách rovinných konstrukcí. (Úloha Apolloniova a úlohy příbuzné.)* JČMF, Cesta k vědění, sv. 4, Prometheus, 1949

[2] Z. Šír: *Approximate parametrisation of confidence sets.* T. Dokken and B. Jüttler, editors, Computational Methods for Algebraic Spline Surfaces, Springer, 2005

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