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ON THE APPLICATIONS OF ORDERED SETS

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Abstract. This paper leads the reader to applications of the mathematics of ordered sets. Our principal goal is to show that the order-theoretical questions widely occur also in the real world and the theoretical results of this branch are interesting and useful for the scientists from many fields. Above all, we will be concerned with the applications of ordered sets to social and computer sciences. We also introduce the basic connections of ordered sets with other mathematical fields, such as topology, matrix algebra and the graph theory.

1. Introduction

The aim of this paper is twofold: First, our principal goal is to discuss relations of the theory of ordered sets to other fields and to demonstrate the usefulness and importance of theoretical results of this branch for practice. In particular, we show that the theory of ordered sets has many applications, not only in mathematics, but also in other areas. Second, we will pay an attention to some of our own results from the theory of ordered sets with respect to their applications. As a matter of fact we show that our results on the number of finite posets, [8] and [9], can be applied in the theory of preferences.

Using such a point of view, we will use slightly popular way of explanation, which should be more accessible to the specialists from various fields. That is why the whole range of notions and relations will be explained by means of suitable examples. It is our belief that this more vivid and illuminative way of exposition should allow easier and more pleasant reading of this work as well as better understanding of all points at issue.

To start with the description of applications of the theory of ordered sets, we have to look back at the beginning of this theory as a branch of mathematics in its own right. In the course of the long historical development people were getting acquainted with many special cases of ordering. First it was arranging and ordering things and phenomena, which one encounters every day. Later people started to acquire habits and skills and this led to the ordering of individual successive steps of some process or some method. Such an ordering of separate steps of a procedure became an acquired experience and was provided to further generations. Starting from the first things and objects, which have been produced or created, people conceived all the time new ideas of arranging and ordering things around and also of ordering their activities. Some of these orderings gradually became self-evident and apparent habits, which we even do not realize. For example, this is the case of the word-order, which arranges the sequence of words in a sentence. Further, we still arrange things according to how we like them. Next, if we want to be successful

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in our activities, then it is necessary to proceed systematically. In other words, it means again to arrange our activity. The study of foreign languages requires the same. Analogously, at a higher level, the organization of all production methods is strictly based on arranging the work. In this area computers became one of the main aiders and control the work of whole factories and enterprises. At the same time, the principle of the work of computers is based on the so-called Boolean method of ordering. On the other hand, in the course of study of the laws of quantum mechanics it turned out to be useful to generalize this type of ordering. The theory of ordered sets, which has been elaborated most of all in the theory of lattices, is now a thriving modern part of contemporary mathematics and has been developing intensively.

During the long historical development there have been formulated the basic properties which characterize the notion of an ordering. Roughly speaking, ordering is certain generalization of the well-known notion of an inequality. The following relational definition of an ordering has been used recently most of the time.

Definition 1.1. A binary relation ρ on a set A is an arbitrary subset of the Cartesian product $A \times A$. The relation ρ is called a partial order or an ordering, if it is endowed with the following three properties.

(1) reflexivity: for all $x \in A, [x, x] \in \rho$,
(2) antisymmetry: for all $x, y \in A$, if $[x, y]$ and $[y, x]$ belong to ρ, then $x = y$,
(3) transitivity: for all $x, y, z \in A$ such that $[x, y]$ and $[y, z]$ belong to ρ, the pair $[x, z]$ also belongs to ρ.

If ρ is an ordering on A, then the ordered pair (A, ρ) will be called a partially ordered set or shortly a poset.

Remark 1.2. In this paper we use the following notation. We write $[x, y] \in \rho$ or $x\rho y$ if x and y are related by ρ and we write $[x, y] \notin \rho$ or $x\rho^c y$ otherwise. Hence $\rho^c = A^2 - \rho$ is the supplement relation to the binary relation ρ. Furthermore, by $\triangle$ we denote the diagonal relation on A, i.e. $\triangle = \{[x, x], x \in A\}$ and by $\exp(A)$ we denote the set of all subsets of A. It is easy to verify that the diagonal relation $\triangle$ gives an example of an ordering relation on A. A set which is ordered by a diagonal relation will be called an antichain. The ordered pair $(\exp(A), \subseteq)$, where $\subseteq$ is a set inclusion, is the basic and very important example of a partially ordered set as well.

In general, our paper is divided into three basic parts. First we shall deal with the connections of ordered sets with other mathematical fields. Then we turn to some applications of partially ordered sets in social sciences, especially to the theory of preferences. In the last section we will be concerned with some applications of partially ordered sets in computer sciences, above all in information systems.

2. The connections of ordered sets with other mathematical fields

The principal aim of this section is to provide some important examples of connections of the ordered sets with other mathematical areas. In particular, we shall deal with the connections of ordered sets with topology, matrix algebra and the graph theory. We point out that the revelation of relations and connections of some branch with other fields has often the practical significance. Indeed, the first advantage of such a connection is that it makes possible an alternative or reversible formulation of a problem. Further, to solve this problem, we can also use the new approach together with the mathematical machinery of other type. It turns out that many times just this heuristic can lead to the solution of an original primary problem, since the reformulation enables us the new point of view. Now we already introduce the first correspondence between ordered sets and the matrix algebra.

Matrix algebra. First we can represent the binary relation ρ on an n-set $\{x_1, \dots, x_n\}$ by means of the matrix $M = (m_{i,j})$ of the type n/n, whose elements are zeros or ones. Then the one-to-one correspondence between such matrices and binary relations is given by

$$m_{i,j} = 1 \Leftrightarrow [x_i, x_j] \in \rho \quad \text{and} \quad m_{i,j} = 0 \Leftrightarrow [x_i, x_j] \notin \rho.$$

Example 2.1. Let us denote $A = \{1, 2, 3, 4\}$ and $\rho = \{[1, 1], [2, 2], [3, 3], [4, 4], [1, 3], [1, 4], [2, 4]\}$. Then we have the following matrix representation M of the binary relation ρ:

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that ρ is a partial order. By the well-known proposition from the linear algebra we have that the determinant of a triangular matrix M in the step form is equal to the product of all members on the main diagonal. In our case it is evident that $\det(M) = 1$. Is it a fortuity or a regularity? In 1971 Kim Ki-Hang Butler [3] has given the answer to this question (see also [4]). He proved the following assertion:

Theorem 2.2. *The matrix M is the matrix representation of a partial order if and only if $M^2 = M$ and $\det(M) = 1$, i.e. M is nonsingular and idempotent.*

Let us remark that the matrix product M^2 in Theorem 2.2 can be computed by means of obvious Boolean operations: $0 + 0 = 0$, $1 + 1 = 1 + 0 = 0 + 1 = 1$, $1.1 = 1$ and $0.1 = 1.0 = 0.0 = 0$. We remark that it is very interesting to observe how the individual properties of an ordering relation are connected with the algebraic properties described in the preceding theorem. Furthermore, the computation with nonsingular idempotent Boolean matrices leads to interesting and difficult questions from the semigroup theory. It is worth mentioning here that especially this problem requires the theory of Green's equivalence relations (cf. [4]).

Graph theory. The second possibility how to look at the partial order has been connected with the graph theory. The binary relation ρ on an n-set $A = \{x_1, \dots, x_n\}$ can be represented by a standard way as a directed graph $G = (V, E)$ with the set of vertices V and the set of edges E. The idea of such a representation is the following. The set of vertices V can be identified with the set A, i.e. $A = V$ and the vertices $x_i, x_j \in V$ are joined by a directed edge iff $[x_i, x_j] \in \rho$. Specially if $[x_i, x_i] \in \rho$, we locate a loop at x_i.

Example 2.3. Represent the binary relation ρ from Example 1 by means of a directed graph. See Figure 1.