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SINGULAR CAUCHY PROBLEM FOR IMPLICIT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. A singular Cauchy problem for a system of implicit Volterra integro-differential equations is considered. With the aid of Banach contraction principle , the theorem concerning the existence and uniqueness of a solution of this problem (having the graph in a prescribed domain) is proved. Moreover, continuous dependence of a solution on a parameter is investigated as well.

Introduction

The problem of the existence and uniqueness of solutions of a singular Cauchy problem for differential and integro-differential equations equations solved with respect to derivatives of unknows has been investigated in detail by many authors (see. e.g. [1-10]). Much attention has also been given to the Cauchy problem for implicit differential and integro-differential equations; both general questions concerning the solvability of this problem and the continuous dependence of solutions on a parameter [11-15] have been considered.

In the present paper, we generalize some results of the paper [10] . A solution of the mentioned implicit system is located in a domain having the vertex coinciding with the initial point.

Consider the following problem:

$$y'(x) = f(x,y,y',\mu) + \int_{0^+}^{x} g(x,s,y(s)y'(s),\mu)\,ds \quad y(0^+,\mu) = 0, \tag{0.1}$$

The functions f, g will be assumed to satisfy:

(I) $f \in C(D_1),\ D_1 = \{(x,u,v,\mu) \in J \times R^n \times R^n \times R : |u| \le \phi(x),\ |v| \le \phi(x),\ J = (0,x_0]\},\ g \in C(D_2),\ D_2 = \{(x,s,u,v,\mu) \in J \times J \times R^n \times R^n \times R : |u| \le \phi(x),\ |v| \le \phi(x)\}$, where $x_0 > 0,\ 0 < \phi(x) \in C^0(J),\ \phi(0^+) = 0$.

(II) $|f(x,\overline{u},\overline{v},\mu) - f(x,\overline{\overline{u}},\overline{\overline{v}},\mu)| \le M_1|\overline{u}-\overline{\overline{u}}| + M_2|\overline{v}-\overline{\overline{v}}|$
for every $(x,\overline{u},\overline{v},\mu),\ (x,\overline{\overline{u}},\overline{\overline{v}},\mu) \in D_1$,
$|g(x,s,\overline{u},\overline{v},\mu) - g(x,s,\overline{\overline{u}},\overline{\overline{v}},\mu)| \le N_1|\overline{u}-\overline{\overline{u}}| + N_2|\overline{v}-\overline{\overline{v}}|$
for every $(x,s,\overline{u},\overline{v},\mu),\ (x,s,\overline{\overline{u}},\overline{\overline{v}},\mu) \in D_2$, where $M_i > 0,\ N_i > 0,\ i = 1,2$

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and $|.|$ denotes the usual norm in R^n.

1. Main results

Theorem 1.1. *If the functions* $f(x,u,v,\mu)$, $g(x,s,u,v,\mu)$ *satisfy* (I),(II) *and*

$$|f(x,u,v,\mu)| \le \Phi(x),$$

where $0 < \Phi(x) \in C(J)$, $\int_{0^+}^x \Phi(s)\,ds \le \alpha\phi(x)$, $\alpha \in R^+$,

$$|g(x,s,u,v,\mu)| \le \psi(x,s),$$

where $0 < \psi(x,s) \in C(J \times J)$, $\int_{0^+}^x \int_{0^+}^s \psi(s,t)\,dt\,ds \le \beta\phi(x)$, $\beta \in R^+$, $\alpha + \beta \le 1$. *Then the initial problem* (0.1) *has a unique solution* $y(x,\mu)$ *for each* $\mu \in R$, $t \in J$.

Proof. Put

$$y(x) = \int_{0^+}^x r(s)ds,$$

where $r(x) \in C^0(J)$ is an unknown function. Then y'(x) = r(x) and system (0.1) is equivalent to the system of integral equations

$$r(x) = \int_{0^+}^x f(s, \int_{0^+}^s r(t)dt, r(s), \mu)\,ds + \int_{0^+}^x \int_{0^+}^s g(s,t,\int_{0^+}^t r(\tau)d\tau, r(t), \mu)\,dt\,ds \tag{1.1}$$

Denote H the Banach space of continuous vector-valued functions

$$h : J_0 \to R^n\ ,\ J_0 = [0, x_0],\ |h(x)| \le \phi(x) \text{ on } J$$

with the norm

$$\|h\|_\lambda = \max_{x \in J_0} \{e^{-\lambda x}|h(x)|\},$$

where $\lambda > 0$ is an arbitrary parameter. Define the operator T by right-hand side of (1.1):

$$T(h) = \int_{0^+}^x f(s, \int_{0^+}^s h(t)dt, h(s), \mu)\,ds + \int_{0^+}^x \int_{0^+}^s g(s,t,\int_{0^+}^t h(\tau)d\tau, h(t), \mu)\,dtds,$$

where $h \in H$. Let $\mu \in R$ be fixed. The transformation T maps H continuously into itself because

$$\begin{aligned}|T(h)| &\le \int_{0^+}^x |f(s, \int_{0^+}^s h(t)dt, h(s), \mu)|\,ds + \int_{0^+}^x \int_{0^+}^s |g(s,t,\int_{0^+}^t h(\tau)d\tau, h(t), \mu)|\,dt\,ds \\ &\le \int_{0^+}^x \Phi(s)\,ds + \int_{0^+}^x \int_{0^+}^s \psi(s,t)\,dt\,ds \le (\alpha+\beta)\phi(x) \le \phi(x)\end{aligned}$$

for every $h \in H$. We shall prove that

$$\|T(h_2) - T(h_1)\|_\lambda \le \left(\frac{M_2}{\lambda} + \frac{M_1 + N_2}{\lambda^2} + \frac{N_1}{\lambda^3}\right) \tag{1.2}$$