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ON PHENOMENON OF RATIONALITY IN GEOMETRIC MODELLING☆

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Dedicated to Prof. Pavel Pech on the occasion of his 60th birthday

Abstract. Exploring the history of geometric modelling (Computer Aided Geometric Design, CAGD) in industrial and related applications we can clearly see that study of rational techniques and rational representations were at its very foundation. However, although CAGD and consequently a vast variety of geometrical applications is based on parametric piecewise rational representations, the major problem is that many natural geometrical operations do not preserve rationality of derived objects (among the most frequent such operations belong offsetting, operation of convolution and Minkowski sums). Hence, studying the rationality belongs to challenging problems of geometric modelling.

Introduction

Geometric modelling or Computer Aided Geometric Design (CAGD) is a research field focused, among others, on the construction and representation of free-form curves, surfaces, or volumes. The history of geometric modelling started in middle 1970s. It was motivated by the necessity to handle more and more complex objects and shapes used e.g. in industry or architecture. Fundamentals of CAGD had already been set up by engineers like Pierre Bézier (see [6, 7]), Paul de Casteljau (see [21, 22]) and Steven Anson Coons (see [17, 18, 19]). Nevertheless, a real development of CAGD waited till modern computers had such sufficient computational power to handle geometrical objects of higher level of complexity. CAGD became a sovereign discipline in its own right after the 1974 conference at the University of Utah. One of the first handbooks devoted to CAGD was published in 1988 (see for instance [26]).

One can find lots of theoretical geometric research fields, such as algebraic geometry (see e.g. [38, 39, 76]), differential projective geometry ([10, 61]), classical sphere and line geometries ([9, 14, 56, 60, 61]) and others, which also deal with curves, surfaces and higher dimensional objects – of course from a slightly different perspective. Hence, CAGD highly benefits from the theoretical results of these disciplines, reformulates and develops these results and corresponding techniques and subsequently incorporates them into various practical applications. And vice versa – investigating applied geometric problems in CAGD can offer many novel and efficient approaches to theoretical geometric disciplines.

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Exploring the history of Computer Aided Geometric Design in industrial and related applications (for more details see [28] and references therein) we can clearly see that rational techniques and rational representations were at the very foundation of geometric modelling. Hence, the history of CAGD is also a history of rational techniques and representations, and vice versa. And going deeper into the past we realize that the phenomenon of rationality shaped the development not only of geometry (and thus geometric modelling) but also of all mathematics.

Let us recall one particular example. If there is any mathematical theorem that is familiar to all educated people it must be the theorem of Pythagoras. It describes a property of right-angled triangles for which it holds that the square of the hypotenuse is equal to the sum of the squares of other two sides. Although Pythagoras' theorem was the first indication of a deeper relation between arithmetic and geometry it brings also some conflicts, followed by the proof that $\sqrt{2}$ is irrational. Mainly, it led to the discovery of quantities which were incommensurable, i.e., not expressible using ratios of whole numbers. The Pythagoreans did not accept $\sqrt{2}$ as a number, which caused the first crisis in the foundations of mathematics (see [69] for more details). However, it is typical for mathematics that elementary "old" knowledge often brings new unexpected results. And thus thousands years later a distinguished property of Pythagorean triples and related di scussions about rationality influenced significantly a development of geometric modelling when curves with Pythagorean hodographs and surfaces with Pythagorean normals were introduced and thus varieties with rational offsets started being thoroughly studied – cf. [33, 59] and the following text fore more details. And this is not, of course, the only example how the demand on rationality and development of CAGD have been closely related.

This article describes shortly the research from recent years, during which various rational techniques were introduced and thoroughly studied. For the sake of brevity, we will focus only on two rational techniques studied in CAGD and used in related applications, namely on offsetting and trimming. The text serves mainly as a summary and detailed discussion of selected key results, most of them already published in distinguished archived journals – hence, see the references at the end of the paper.

1. Rationality and representation of shapes in CAGD

The choice of an appropriate representation of geometric objects (explicit, parametric, or implicit one) is a important issue for the development of consequent algorithms. Whereas for instance computer graphics use all the above mentioned representations, CAD focuses only on some of them – mainly on the parametric one. Among all parameterizations, the most important ones are those that can be described with the help of polynomials or rational functions since these representations can be easily included into standard CAD systems and then used in technical applications. Parameterizations allow us to generate points on curves and surfaces, they are also suitable for surface plotting, computing transformations, determining offsets, computing curvatures e.g. for shading and colouring, for surface-surface intersection problems etc. On the other hand implicit representations are suitable e.g. to decide whether a given point is lying on the curve or surface.

To give a simple example, let us consider the unit sphere given by the implicit equation $\mathcal{S} : x^2 + y^2 + z^2 - 1 = 0$ which possesses the well known parameterization

$$\mathbf{x}(s,t) = \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1} \right)^{\top}, \quad s,t \in \mathbb{R} \tag{1.1}$$

obtained by the stereographic projection. However, not every algebraic curve or surface admits a rational parameterization. So, suitable parameterization algorithms are necessary to decide whether a parameterization exists (or not) and produce one in the positive case. The reverse problem (consider a rational parametric description, find the corresponding implicit equation) is called the implicitization problem.

To be more exact. Let $\mathcal{V}$ be a variety of dimension d over a field $\mathbb{K}$. Then $\mathcal{V}$ is said to be *unirational*, or *parametric*, if there exists a rational map $\mathcal{P} : \mathbb{K}^d \to \mathcal{V}$ such that $\mathcal{P}(\mathbb{K}^d)$ is dense in $\mathcal{V}$. We speak about a (*rational*) *parameterization* $\mathcal{P}(t_1,\dots,t_d)$ of $\mathcal{V}$. Furthermore, if $\mathcal{P}$ defines a birational map then $\mathcal{V}$ is called *rational*, and we say that $\mathcal{P}(t_1,\dots,t_d)$ is a *proper parameterization*. Problems originated in geometric modelling work especially over the reals and thus some difficulties may appear and certain algebraic geometry techniques (developed for algebraically closed fields) must be reconsidered when used in particular applications. Let us recall results at least for algebraic curves in plane and surfaces in space.

By a theorem of Lüroth, a curve has a parameterization iff it has a proper parameterization iff its genus (see [68] for a definition of this notion) vanishes. Hence, for planar curves the notions of rationality and unirationality are equivalent for any field. Algorithmically, the parameterization problem is well-solved, see e.g. [8, 63, 67, 72, 75, 77]. In the surface case, the theory differs as Castelnuovo's theorem holds only for algebraically closed fields of characteristic zero. By this theorem, a surface has a parameterization iff it has a proper parameterization iff the arithmetical genus p_a and the second plurigenus P_2 are both zero (see [68] for a definition of these notions). The problem is algorithmically much more difficult than for curves – see e.g. [1, 55, 66, 64, 65, 70] for further details.

Today, the NURBS representation (where NURBS stands for Non-Uniform Rational B-Spline) is considered as a universal standard in technical praxis, which offers a unifying exchanging data format and enables to represent e.g. basic spline curves, conics, quadrics and many other elementary geometric objects from technical applications. Due to space limitations we recall in this section only *Non-Uniform Rational B-Spline* (NURBS) curves and surfaces and rational triangular Bézier surfaces. The reader interested in more details, especially in thorough discussion of their properties, is kindly referred to [27, 58, 61].

Let us look firstly at the curve case. NURBS curves are generalizations of both B-splines and Bézier curves mentioned above, as polynomial B-splines can be decomposed into polynomial Bézier curves and the weights of the control points makes curves being rational. Consider an integer k (the *degree*), and an ordered list T (the *knot vector*) of real numbers $t_0 \le t_1 \le \ldots \le t_m$. We define recursively functions $N_i^k(t)$ as follows

$$\begin{aligned} N_i^0(t) &:= \begin{cases} 1 & \text{for} \quad t_1 \le t < t_{i+1}, \\ 0 & \text{otherwiese}, \end{cases} \\ N_i^r(t) &:= \frac{t-t_i}{t_{i+r}-t_i} N_i^{r-1}(t) + \frac{t_{i+r+1}-t}{t_{i+r+1}-t_{i+1}} N_{i+1}^{r-1}(t) \end{aligned} \tag{1.2}$$

for $1 \le r \le k$. $N_i^k(t)$ is called the i-th *B-spline basis function* of degree k corresponding to the knot vector T. It can be proved that Bernstein polynomials (describing Bézier curves) can be considered as a special B-spline basis.

If k is a positive number, $\mathbf{b}_0, \dots, \mathbf{b}_n$, $k \le n$, are so called *control points* in $\mathbb{R}^d$ (especially $d = 2,3$) with the corresponding *weights* $w_0, \dots, w_n$ and T is a knot vector, then the associated *NURBS curve* is defined by

$$\mathbf{c}(t) = \frac{\sum_{i=0}^n w_i N_i^k(t) \mathbf{b}_i}{\sum_{i=0}^n w_i N_i^k(t)}. \tag{1.3}$$

Clearly, the control points determine the shape of the curve as each point of the curve is computed by taking a weighted sum of the control points. It holds that all rational curves can be written in the NURBS form (1.3).

A simple and straightforward generalization of the notion NURBS curve are *tensor product NURBS surfaces*. These surfaces have the form

$$\mathbf{s}(u,v) = \frac{\sum_{i=0}^n \sum_{j=0}^m w_{ij} N_i^k(u) N_j^\ell(v) \mathbf{b}_{ij}}{\sum_{i=0}^n \sum_{j=0}^m w_{ij} N_i^k(u) N_j^\ell(v)}, \tag{1.4}$$

where $\mathbf{b}_{ij}$ are control points with the associated weights w_{ij} and $N_i^k(u)$, or $N_j^\ell(v)$ are the standard B-spline basis functions defined by a certain knot vectors $u_0 \le u_1 \le \dots u_{n+k+1}$, or $v_0 \le v_1 \le \dots \le v_{m+\ell+1}$, respectively.

NURBS curves and surfaces are today very popular for a number of reasons which make them one of the most widespread representations used in various applications. We recall at least some of them:

- they can be evaluated quickly;
- they offer one common mathematical form for both standard analytical shapes (e.g. conics or quadrics) and free-form shapes used e.g. in industry or architecture;
- they provide the flexibility to design a large amount of shapes;
- they are invariant under projective transformations, i.e., operations like rotations, translations, affinities, or perspectives can be applied to NURBS curves or surfaces by applying them to their control points;
- they reduce the memory consumption when storing shapes.

In addition, we recall another wide-spread type of parametric representation of rational surfaces in CAGD. Any rational parametric surface of degree n can be described by the so called *triangular Bernstein–Bézier representation*

$$\mathbf{s}(u,v,w) \frac{\displaystyle\sum_{i,j,k \in \mathbb{Z}^+,\ i+j+k=n} w_{ijk}\, \mathbf{b}_{ijk}\, \frac{n!}{i!j!k!} u^i v^j w^k}{\displaystyle\sum_{i,j,k \in \mathbb{Z}^+,\ i+j+k=n} w_{ijk}\, \frac{n!}{i!j!k!} u^i v^j w^k}, \tag{1.5}$$

where $u + v + w = 1$. The shape of the surface is determined by the $\binom{n+1}{2}$ control points $\mathbf{b}_{ijk}$ with the associated weights w_{ijk}. If we set $w = 1 - u - v$ we obtain a power basis representation with the parameters u, v varying within a certain domain triangle $\triangle \subset \mathbb{R}^2$.

We would like to emphasize that although Computer Aided Geometric Design, and consequently a vast variety of geometrical applications, is often based on parametric piecewise rational representations, the major problem of CAGD is that many natural geometrical operations do not preserve rationality of derived objects.

Among the most frequent such operations belong offsetting, operation of convolution and construction of Minkowski sums. Hence, studying their rationality belongs to challenging problems of geometry and geometric modelling.

2. Rational offsets to rational curves and surfaces

Offsetting is one of the fundamental operations in Computer Aided Design and related practical applications (e.g. numerical-control machining, robot path-planning, tolerance analysis). The two-sided δ-offset $\mathcal{O}_\delta(\mathcal{X})$ to an irreducible hypersurface $\mathcal{X}$ is the envelope of the system of circles/spheres centered at the points of $\mathcal{X}$ with the radius δ. A main problem with offset curves and surfaces is that these varieties are often far more complicated than the associated generating objects – e.g. offsets to rational hypersurfaces are nor rational, in general. Even in the case of planar algebraic curves, studying offsets leads to important and challenging computational problems. For more details see e.g. [3, 4, 24, 25, 29, 49, 57].

On the other hand, offsets to certain special classes of curves and surfaces admit exact rational representations. In the case of planar curves, the class of Pythagorean Hodograph (PH) curves as polynomial curves possessing rational offset curves and polynomial arc-length functions was introduced in [33]. A thorough analysis of PH curves has followed – see e.g. [27, 32, 40, 51, 73]. Later, the concept of polynomial planar PH curves was generalized to space PH curves ([31, 34, 35, 37, 74]) and to rational PH curves ([56, 59, 60]). Analogously, the notion of rational surfaces with rational offsets, the so called Pythagorean Normal vector (PN) surfaces, was introduced in [59]. More details about PN surfaces can be found in [43, 45, 46, 48]. For a survey of shapes with Pythagorean normals property (i.e., possessing rational offsets) see [30].

Given a regular C^1 parametric curve $\mathbf{x}(t) = (x_1(t), x_2(t))^\top$, the *offset* of $\mathbf{x}(t)$ is the set of all points in $\mathbb{R}^2$ that lie at a perpendicular distance δ from $\mathbf{x}(t)$. The two branches of the offset are given by

$$\mathbf{x}_\delta(t) = \mathbf{x}(t) \pm \delta\,\mathbf{n}(t), \qquad \mathbf{n}(t) = \frac{\mathbf{x}'(t)^\perp}{\|\mathbf{x}'(t)\|}, \tag{2.1}$$

where $\|\mathbf{x}'(t)\| = \sqrt{x_1'(t)^2 + x_2'(t)^2}$ and $\mathbf{x}'(t)^\perp = (-x_2'(t), x_1'(t))^\top$, i.e., $\mathbf{v}^\perp$ denotes the rotation of $\mathbf{v} \in \mathbb{R}^2$ about the origin by the angle $\frac{\pi}{2}$.

Offset curves are used mainly in numerically controlled machining. They describe the trajectory of a round cutting tool, which is parallel to the cut by a constant distance in the direction normal to the cut at every point. However, even for rational $\mathbf{x}(t)$ the rationality of its offsets is generally not guaranteed. A study of offset rationality led to the class of planar *Pythagorean hodograph* (PH) curves. These curves are defined as rational curves $\mathbf{x}(t) = (x_1(t), x_2(t))^\top$ fulfilling the distinguishing condition

$$\mathbf{x}'(t) \cdot \mathbf{x}'(t) = x_1'(t)^2 + x_2'(t)^2 = \sigma(t)^2, \tag{2.2}$$

where $\sigma(t)$ is a rational function and '$\cdot$' is the standard Euclidean inner product. Since the rationality of a δ-offset curve $\mathbf{x}_\delta(t)$ of a rational curve only depends on the rationality of the unit normal field $\mathbf{n}(t)$, cf. (2.1), planar PH curves posses (piece-wise) rational offsets.

Pythagorean hodograph curves were originally introduced by [33] as *planar polynomial* curves. It was proved [33, 44] that the coordinates of hodographs of polynomial PH curves and $\sigma(t)$ form the following Pythagorean triples

$$
\begin{array}{rcl}
x_1'(t) & = & w(t)\big(u^2(t)-v^2(t)\big),\\
x_2'(t) & = & 2w(t)u(t)v(t),\\
\sigma(t) & = & w(t)(u^2(t)+v^2(t)),
\end{array}
\tag{2.3}
$$

where $u(t)$, $v(t)$, $w(t)\in\mathbb{R}[t]$ are any non-zero polynomials and $u(t),v(t)$ are relatively prime.

A generalization of polynomial PH curves to rational ones was introduced and studied in [59]. This approach uses the dual representation of a plane curve considered as an envelope of its tangents

$$
T: n_1(t)x_1+n_2(t)x_2-h(t)=0,\qquad n_1(t),n_2(t),h(t)\in\mathbb{R}(t).
\tag{2.4}
$$

In order to guarantee the rationality of (2.1), the unit normal field $\mathbf{n}(t)$ must rationally parameterize the unit circle $\mathcal{S}^1$. Hence, there must exist relatively prime polynomials $k(t),l(t)$ such that

$$
n_1(t)=\frac{2k(t)l(t)}{k^2(t)+l^2(t)},\quad n_2(t)=\frac{k^2(t)-l^2(t)}{k^2(t)+l^2(t)}.
\tag{2.5}
$$

In addition, to simplify further computations we set $g=h(k^2+l^2)$, i.e., the dual representation of any arbitrary PH curve is

$$
(2kl)x_1+(k^2-l^2)x_2-g=0.
\tag{2.6}
$$

Consequently, a parametric representation of all planar rational PH curves is obtained as the envelope of their tangents given by (2.6) in the form

$$
x_1=\frac{2(ll'-kk')g+(k^2-l^2)g'}{2(k^2+l^2)(kl'-k'l)},\quad x_2=\frac{(k'l+kl')g-klg'}{(k^2+l^2)(kl'-k'l)}.
\tag{2.7}
$$

Furthermore, the representation of offsets can be easily obtained by translating the tangents by a distance δ, i.e., it is sufficient to replace $g(t)=h(t)(k(t)^2+l(t)^2)$ by $g(t)=(h(t)\pm\delta)(k(t)^2+l(t)^2)$ in (2.7).

Analogously for the surface case, consider a regular C^1 surface $\mathbf{x}:\mathbb{R}^2\to\mathbb{R}^3$ given by a parameterization $\mathbf{x}(u,v)$. The *δ-offset* of $\mathbf{x}(u,v)$ is the set of all points in $\mathbb{R}^3$ that lie at a distance δ from $\mathbf{x}$. The two branches of the offset are given by

$$
\mathbf{x}_\delta(u,v)=\mathbf{x}(u,v)\pm\delta\,\mathbf{n}(u,v),\qquad \mathbf{n}(u,v)=\frac{\mathbf{x}_1\times\mathbf{x}_2}{\|\mathbf{x}_1\times\mathbf{x}_2\|},
\tag{2.8}
$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are partial derivatives with respect to u and v, respectively.

A study of offset rationality led to the class of *surfaces with Pythagorean normal vector fields* introduced in [59]. We use a modified description (of course, equivalent with the original one). Let $\mathbf{x}(u,v)$ be a rational surface (generally in $\mathbb{R}^n$, $n=3,4,\ldots$) for which there exists a rational function $\sigma(u,v)\in\mathbb{R}(u,v)$ such that

$$
G(u,v)=\det(g_{ij})=g_{11}g_{22}-g_{12}^2=EG-F^2=\sigma(u,v)^2,
\tag{2.9}
$$

where $G(u,v)$ is the associated Gram determinant with the entries $g_{ij}=\mathbf{x}_i\cdot\mathbf{x}_j$ and $g_{11}=E$, $g_{12}=F$, $g_{22}=G$ are the coefficients of the first fundamental form. As the Gramian has for surfaces in $\mathbb{R}^3$ (i.e., being hypersurfaces) also the following form

$$
G(u,v)=g_{11}g_{22}-g_{12}^2=\|\mathbf{x}_1\times\mathbf{x}_2\|^2,
\tag{2.10}
$$

then condition (2.9) shows that surfaces with *Pythagorean area elements* are in $\mathbb{R}^3$ equivalent to PN surfaces. Hence, (2.9) guarantees rational offsets of $\mathbf{x}$, cf. (2.8).

For the sake of brevity, we will deal only with non-developable PN surfaces. Such surfaces $\mathbf{x}(u,v)$ can be obtained as the envelope of a two-parametric set of the associated tangent planes

$$T(u,v): \quad \mathbf{n}(u,v)\cdot\mathbf{x}(u,v) - h(u,v) = 0, \tag{2.11}$$

where $h(u,v) = e(u,v)/f(u,v)$ is a rational function (the so called *support function*) representing the oriented distance from the origin and $\mathbf{n}(u,v)$ is a rational parameterization of the unit sphere $\mathcal{S}^2$, cf. [59],

$$\mathbf{n}(u,v) = \left(\frac{2ac}{a^2+b^2+c^2}, \frac{2bc}{a^2+b^2+c^2}, \frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)^\top, \tag{2.12}$$

with $a = a(u,v)$, $b = b(u,v)$, $c = c(u,v)$ fulfilling the condition $\gcd(a,b,c) = 1$. A parametric representation of an arbitrary non-developable PN surface can be then obtained from the system of equations $T = 0$, $\partial T/\partial u = 0$, $\partial T/\partial v = 0$ using Cramer's rule – see formula (3.3) in [59]. Furthermore, the representation of offsets can be again easily obtained by translating the tangent planes by a distance δ, i.e., it is sufficient to replace $h(t)$ by $h(t) \pm \delta$ in (2.11).

Finally, we would like to emphasize one important fact. The interplay between the different approaches to polynomial and rational PH curves was thoroughly studied in [36] and the former were established as a proper subset of the latter by presenting simple algebraic constraints. On the other hand, the situation in the surface case is absolutely different. We have only a general description of rational Pythagorean normal vector surfaces reflecting again the dual approach and polynomial formulae (including an algebraic condition for the specialization of rational PN surfaces to polynomial ones) have not been known yet. Hence, finding a polynomial solution of the Pythagorean condition in the surface case still remains an open and challenging problem – only some partial results can be found e.g. in [46, 71].

3. Offsetting and trimming inseparable

Even though PH curves and PN surfaces admit rational offsets, the usually most demanding part in practice, the *trimming*, still needs to be considered as in the case of approximation techniques. An alternative approach to the problem in the curve case based on the medial axis transform (MAT) of a planar domain, introduced in [11], was formulated in [15, 50].

Consider a planar domain $\Omega \subset \mathbb{R}^2$ and the family of all inscribed discs in Ω partially ordered with respect to inclusion. An inscribed disc is called maximal if it is not contained in any other inscribed disc. Then the *medial axis* $\mathrm{MA}(\Omega)$ is the locus of all centers $(y_1,y_2)^\top$ of maximal inscribed discs and the *medial axis transform* $\mathrm{MAT}(\Omega)$ is obtained by appending the corresponding disc radii y_3 to the medial axis, i.e., MAT consists of points $\mathbf{y} = (y_1,y_2,y_3)^\top$. The projection

$$\mathbb{R}^{2,1} \to \mathbb{R}^2: \quad \mathbf{y} = (y_1,y_2,y_3)^\top \mapsto \overset{\triangledown}{\mathbf{y}} = (y_1,y_2)^\top \tag{3.1}$$

naturally relates MA to MAT (read $\overset{\triangledown}{\mathbf{y}}$ as '$\mathbf{y}$ down').

For a C^1 segment $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))^\top$ of MAT(Ω) we can compute the corresponding boundary of Ω from the envelope formula, cf. [16, 50], in the form

$$\mathbf{x}_\pm(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \frac{y_3}{{y_1'}^2 + {y_2'}^2}\left[y_3' \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \pm \sqrt{{y_1'}^2 + {y_2'}^2 - {y_3'}^2} \begin{pmatrix} -y_2' \\ y_1' \end{pmatrix} \right]. \tag{3.2}$$

A study of rationality of envelopes (3.2) led to the class of *Minkowski Pythagorean hodograph* (MPH) curves introduced in [50]. MPH curves are defined as rational curves $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))^\top$ in three-dimensional space fulfilling the condition

$${y_1'}^2(t) + {y_2'}^2(t) - {y_3'}^2(t) = \varrho^2(t), \tag{3.3}$$

where $\varrho(t) \in \mathbb{R}(t)$. The PH condition (2.2) is now fulfilled with respect to the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3. \tag{3.4}$$

A three-dimensional real affine space along with the indefinite bilinear form (3.4) is called *Minkowski space* and denoted $\mathbb{R}^{2,1}$.

As discussed in [16] and [50], if MAT(Ω) is an MPH curve $\mathbf{y}$, then the boundary curves $\mathbf{x}_\pm$ of Ω associated with $\mathbf{y}$ and all offsets of the boundary are (piece-wise) rational, cf. (3.2). We rewrite (3.2) into the form

$$\mathbf{x}_\pm = \overset{\triangledown}{\mathbf{y}} - y_3 \mathbf{n}_\pm, \tag{3.5}$$

where

$$\mathbf{n}_\pm = \frac{1}{\varrho^2 + y_3'^2}\begin{pmatrix} y_3' y_1' \mp \varrho y_2' \\ y_3' y_2' \pm \varrho y_1' \end{pmatrix} \frac{1}{\varrho^2 + y_3'^2}(y_3' \overset{\triangledown}{\mathbf{y}}{}' \mp \rho \overset{\triangledown}{\mathbf{y}}{}'^\perp). \tag{3.6}$$

It can be shown by a direct computation that $\mathbf{n}_\pm$ is a unit vector perpendicular to $\mathbf{x}_\pm$. Moreover, $\mathbf{n}_\pm$ is rational if and only if ϱ is rational. Hence, for any MPH curve $\mathbf{y} \subset \mathbb{R}^{2,1}$, the *associated* curves $\mathbf{x}_\pm \subset \mathbb{R}^2$ possess a normal vector field rationally parameterizing the unit circle, i.e., $\mathbf{x}_\pm$ are rational PH curves. We recall the result from [42] that any rational MPH curve $\mathbf{y}$ in $\mathbb{R}^{2,1}$ can be constructed starting from an (associated) planar rational PH curve $\mathbf{x}$ in $\mathbb{R}^2$ and a rational function r in the form

$$\mathbf{y}(t) = (x_1 + r n_1, x_2 + r n_2, r)^\top = \overset{\vartriangle}{\mathbf{x}}(t) + r(t)\tilde{\mathbf{n}}(t), \tag{3.7}$$

with $\overset{\vartriangle}{\mathbf{x}} = (x_1, x_2, 0)^\top$ (read $\overset{\vartriangle}{\mathbf{x}}$ as '$\mathbf{x}$ up') and $\tilde{\mathbf{n}} = (n_1, n_2, 1)^\top$, where $\mathbf{n} = (n_1, n_2)^\top = \mathbf{x}'^\perp / \sigma$.

Using (3.7) and (2.7), we can show that a curve $\mathbf{y} \in \mathbb{R}^{2,1}$ is an MPH curve if and only if there exist two polynomials k, l and two rational functions g, r such that

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \frac{1}{2(k^2+l^2)(kl' - k'l)} \begin{pmatrix} 2(ll' - kk')g + (k^2 - l^2)g' \\ 2(k'l + kl')g - 2klg' \\ 0 \end{pmatrix} + \frac{r}{k^2 + l^2}\begin{pmatrix} 2kl \\ k^2 - l^2 \\ k^2 + l^2 \end{pmatrix}. \tag{3.8}$$

Obviously, *polynomial* MPH curves form a proper subset of the set of rational MPH curves described by (3.8). It turns out that we can efficiently adapt the approach for relating planar rational and polynomial PH curves used in [36] and show that any polynomial MPH curve in $\mathbb{R}^{2,1}$ can be obtained using (3.8) by setting

$$\begin{aligned} g(t) &= 2kl \int (km - ln)\,\mathrm{d}t - (k^2 - l^2)\int (kn + lm)\,\mathrm{d}t - (k^2 + l^2)\int (lm - kn)\,\mathrm{d}t, \\ r(t) &= \int (lm - kn)\,\mathrm{d}t, \end{aligned} \tag{3.9}$$

where $k(t), l(t), m(t), n(t)$ are arbitrary polynomials.

Substituting (3.9) into (3.8) we obtain an alternative to the original formula for polynomial MPH curves presented in [50]

$$y_1' = km - ln, \quad y_2' = -kn - lm, \quad y_3' = -kn + lm, \quad \varrho = km + ln. \tag{3.10}$$

The situation in three-dimensional space has become an active research area recently, since the so called MOS surfaces as a spatial analogy of MPH curves were introduced in [41]. The medial surface transform (MST) of a volume is the set of surface patches (or curve segments) in four-dimensional Minkowski space $\mathbb{R}^{3,1}$ such that each point of these surfaces/curves represents the center and the radius of a maximal ball inscribed into the domain. The MST covers the structure of the domain and thus can be used in all sorts of geometric modelers (constructive solid geometry (CSG) and boundary representation (B-rep)) as a useful tool. The distinguishing property of MOS surfaces is that if considered as the MST of a volume, the associated envelope and its offsets admit exact rational parameterization. Later, it was proved in [54] that quadratic triangular Bézier surfaces in $\mathbb{R}^{3,1}$ possess the MOS property and a related study followed in [4, 5, 52, 53]. However, for the sake of brevity we omit this part – the reader interested in this topic is kindly referred to the literature.

Finally, we would like to emphasize that using the MAT/MST representation makes the trimming procedure for the inner offsets very simple – only those parts of the MAT/MST where the corresponding circle/sphere radius is less than the offset distance δ have to be trimmed. For more details see e.g. [2, 4, 12, 13, 20, 23, 47, 62].

Conclusion

This paper gave a short overview of some rational techniques which have been studied during the recent years. The study was motivated by the fact that most of geometrical operations used in Computer Aided Geometric Design do not preserve rationality of derived objects in general. Of course, this article cannot cover every related aspect yet it represents our best effort to provide a comprehensive description of knowledge that has been collected to date. We believe that it can also help another purpose, i.e., to show that studying rational techniques for geometric modelling (and related applications) is a beautiful and active research area which provides fascinating and challenging mathematical problems on the edge of basic and applied research.

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