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POTENTIAL OF THE "ENDS LEMMA" TO CREATE RING-LIKE HYPERSTRUCTURES FROM QUASI-ORDERED (SEMI)GROUPS

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ABSTRACT. The article deals with the hyperstructure theory. In 1990's CHVALINA presented a way of creating semi-hypergroups and hypergroups from partially / quasi–ordered semigroups and groups (called the "Ends lemma"). In this article I study its extension in the theory of rings and hyperrings.

1. Motivation

A number of articles and contributions in the hyperstructure theory (especially by Czech authors such as CHVALINA, CHVALINOVÁ, HOŠKOVÁ, RAČKOVÁ, MOUČKA or NOVÁK) make use of the construction first used in [4] as Theorems 1.3 and 1.4 (chapter IV), pp. 146–147. Using these results known as the "Ends lemma" we can form hyperstructures from quasi / partially ordered structures. For examples cf. e.g. [5, 6, 8, 13]. The conditions for creating semihypergroups and hypergroups (or rather transposition hypergroups or join spaces) have already been established. However, the potential of the "Ends lemma" to create hyperstructures with two (hyper)operations has not been studied yet. In this article I discuss a simple question: *Roughly speaking, using the "Ends lemma" semigroups create semihypergroups and groups create hypergroups. Is it true that rings create hyperrings?* The answer is rather complex as the concept of a hyperring is a broad one and permits a number of generalizations and special cases.

2. Preliminaries

Recall first some basic definitions and ideas from the hyperstructures theory. A *hypergrupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \to \mathcal{P}^*(H)$ is a binary hyperoperation on H. Symbol $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom: for any element $a \in H$ equalities $a \bullet H = H = H \bullet a$ hold, is satisfied, then the pair $(H, \bullet)$ is called a *hypergroup*. A hypergroup $(H, \bullet)$ is called a *transposition hypergroup* if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a \bullet d \approx b \bullet c$, where $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$. Sets $b\backslash a = \{x \in H; a \in b \bullet x\}$ and $c/d = \{x \in H; c \in x \bullet d\}$ are called

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left and *right extensions*, or *fractions*, respectively. A commutative transposition hypergroup is called a *join space*.

An element of $e \in H$, where $(H, \bullet)$ is a hyperstructure, is called an *identity* if for $\forall x \in H$ there holds $x \bullet e \ni x \in e \bullet x$. If for $\forall x \in H$ there holds $x \bullet e = \{x\} = e \bullet x$, then $e \in H$ is called a *scalar identity*.

As far as the theory of ordered structures is concerned, we need to recall that by a *quasi-ordered (semi)group* we mean a triple $(G, \cdot, \leq)$, where $(G, \cdot)$ is a (semi)group and $\leq$ is a reflexive and transitive binary relation on G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold. This condition is equivalent to stating that for every quadruple of elements $a, b, c, d \in G$ such that $a \leq b$, $c \leq d$ there holds $a \cdot c \leq b \cdot d$. We call the semigroup *partially ordered*[1] if the relation $\leq$ is moreover antisymmetric. Finally, $[a)_\leq = \{x \in G; a \leq x\}$ is a *principal end* generated by $a \in G$.

As far as the ring theory is concerned, recall that by a *ring* we mean a structure $(H, +, \cdot)$ such that $(H, +)$ is a group, $(H, \cdot)$ is a semigroup and $\cdot$ distributes over $+$ from both left and right. From these axioms there follows that the neutral element $0 \in H$ has the absorbing property with respect to multiplication, i.e. that there holds $0 \cdot a = a \cdot 0 = 0$ for an arbitrary $a \in H$. A *semiring* is a structure $(H, +, \cdot)$ such that both $(H, +)$ and $(H, \cdot)$ are semigroups or monoids,[2] multiplication distributes over the addition $+$ from both left and right and the absorbing property of the neutral element of $(H, +)$ with respect to multiplication is postulated. Respective definitions from the hyperring theory will be included later in section 3.

We are going to examine the "Ends lemma", which has the form of the following Theorems:

Theorem 2.1. *([4], Theorem 1.3, p. 146) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}'(S)$ defined by*

$$a * b = [a \cdot b)_\leq$$

is associative. The semi-hypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

In accordance with [11], the hyperstructure $(S, *)$ constructed in this way will further on be called the *associated* hyperstructure to the structure $(S, \cdot)$ or *an "Ends lemma"–based hyperstructure*. Instead of S the carrier set will be denoted by H.

Theorem 2.2. *([4], Theorem 1.4, p. 147) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

- 1^0**:** *For any pair $a, b \in S$ there exists a pair $c, c' \in S$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$*
- 2^0**:** *The associated semi-hypergroup $(S, *)$ is a hypergroup.*

Remark 2.3. If $(S, \cdot, \leq)$ is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S, \cdot, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.

Remark 2.4. The wording of the above Theorems is the exact translation of theorems from [4]. The respective proofs, however, do not change in any way, if we regard *quasi-ordered* structures instead of *partially ordered* ones as the anti-symmetry

[1] In fact, the term *ordered* is often used in the Czech environment by authors such as Chvalina, Hošková, Račková or even myself instead of the correct English term *partially ordered*.

[2] The fact whether $(H, \cdot)$ has the neutral element makes no difference in this article.

of the relation $\leq$ is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi–ordered structures.

The following theorem extending the "Ends lemma" was proved by Račková in her Ph.D. thesis. The proof can be also found in [13]. Notice that if $(H, \cdot)$ is commutative, then $(H, *)$ is a join space.

Theorem 2.5. *([13], Theorem 4) Let $(H, \cdot, \leq)$ be a quasi-ordered group and $(H, *)$ be the associated hypergroupoid. Then $(H, *)$ is the transposition hypergroup.*

Finally, notice the main result of article [11] and especially its immediate corollary – the fact that $(H, *)$ is not a canonical hypergroup.

Theorem 2.6. *([11], Theorem 3.1) Let $(H, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(H, *)$ be its associated transposition hypergroup. Then $(H, *)$ does not have a scalar identity.*

3. Intention

There exists a number of hyperstructure analogies of single-valued structures with two operations: as far as rings and their generalizations are concerned, there exist hyperrings in the general sense, (Krasner) hyperrings, semihyperrings, hyperringoids, hypernearrings, etc. In this section, I am going to study the potential of "Ends lemma"–based hyperstructures to create these structures. For details and respective definitions cf. e.g [2, 7, 9, 14] or Definition 5.1 or Definition 5.7.

There are three evident ways of creating a hyperstructure analogy of single-valued structures with two operations:

(1) Let $(H, +)$ and $(H, \cdot)$ be two single-valued structures. We can define a hyperoperation using one of the operations $+$ or $\cdot$ by e.g. $a * b = [a + b)_\leq$ – thus we get an "Ends lemma" based hyperstructure $(H, *)$. The hyperstructure will then be a triplet $(H, *, \cdot)$ where $*$ is a hyperoperation based on the single-valued operation $+$.

(2) Let $(H, +)$ and $(H, \cdot)$ be two single-valued structures. We can define two hyperoperations, each based on one single-valued operation, i.e. for an arbitrary pair $a, b \in H$ we can define $a * b = [a + b)_\leq$ and $a \circ b = [a \cdot b)_\leq$. Thus we get a triplet $(H, *, \circ)$, where $*$ and $\circ$ are hyperoperations.

(3) However, we can also start with a single single-valued structure $(H, \cdot)$ and using it define a hyperoperation $*$ by $a * b = [a \cdot b)_\leq$. The hyperstructure will then be a triplet $(H, *, \cdot)$ where $*$ is a hyperoperation based on the single-valued operation $\cdot$.

If successful, idea 2 will result in *hyperrings in the general sense* or a *semihyperring* as defined in e.g. [7] or [14] while ideas 1 and 3 will result in *additive / multiplicative hyperrings*, *Krasner hyperrings*, *hyperringoids*, *semihyperrings* in the recent sense (i.e. with one hyperoperation and one single-valued operation) as used in recent works by Davvaz, works by Ameri or in [3] or *hypernearrings*.

As far as "Ends lemma"–based hyperstructures are concerned, we know that semihypergroups are created from semigroups, hypergroups are created from groups, or rather semigroups with a special property, and transposition hypergroups are created from groups. We also know that commutative single-valued structures create commutative hyperstructures. However, Theorem 2.6 states that no "reasonable"