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RATIOS AND MEAN VALUES: A TOPIC IN ART, ARCHITECTURE AND MATHEMATICS

GUNTER WEISS

Abstract. The paper deals with some old and new Mean Values based on the properties and the treatment of the Golden Mean as fundamental example. The less known mean values, *van der Laan*'s number ψ and *Rosenbusch*'s number ρ, are treated in more detailed. Hereby graphical constructions and paper folding operations (origami) will show, that the topic is well suited for maths-educational purposes as well as it opens up for an actual mathematical research field with applications in arts and architecture.

1. Introduction

Same decades ago the Dutch Benedictine monk and architect Dom Hans van der Laan (1904-1991) proposed a new ratio, related to the Golden Ratio, as basic module ψ in architectural design. More recently, the German architect Lambert Rosenbusch (1940-2009) invented an analogue module, the "cubi ratio" ρ and he even used a construction of it as his personal logo. From the mathematical viewpoint both numbers, ψ as well as ρ, are solutions of cubic equations, which both are somehow natural generalizations to the quadratic equation of the Golden Mean. So it is obvious to treat these new Mean Values based on the properties and the treatment of the Golden Mean as the fundamental example. As irreducible cubic equations cannot be solved with ruler and compass operations alone, one might look for procedures allowing the graphical treatment of cubic equations, too, and one finds such an elementary constructive procedure in applying "mathematical origami". This approach makes the topic well suited for maths-educational purposes as well as it opens up for an actual mathematical research field with applications in arts and architecture.

While the first chapter collects properties and constructions of the Golden Mean φ and the Metallic Means (see [9], [2] and [11]) and it gives a projective geometric property of φ. The next chapter presents van der Laan's number ψ (see [9]) and Rosenbusch's number ρ (see [8]). Finally their properties and constructions are compared with those of classical cubic problems, angle trisection, doubling of the cube and constructing a regular hexagon.

2. Visualizing elementary number theory

Stefan Deschauer [2] posed the following question:

(2.1) $$\text{Find all pairs } (x, y) \text{ of digits with } x + \frac{y}{10} = \frac{y}{x}.$$

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Obviously $(2,5)$ is such a pair of digits, and for decimal fractions it is easy to see that there is no other solution. But one can generalize the condition (2.1) by using another decimal basis g:

$$\frac{y}{x} = x + q^{-1}y, \qquad q := g^k; \qquad g, k \in \mathbb{N}, \qquad 2 \leq x < y < q \tag{2.2}$$

While for $x,\ y \in \square$ the left equation describes a plane π, the right equation describes a special hyperbolic paraboloid Φ, such that all solutions belong to the intersection $c := \pi \cap \Phi$ and to the integer grid in $\square^2$, see Fig. 1.

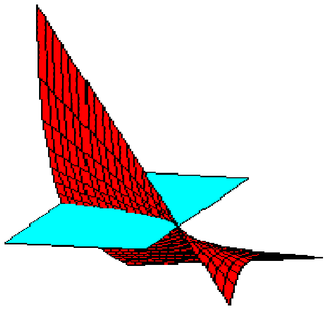

FIGURE 1. Deschauer's problem interpreted in space as intersection of a hyperbolic paraboloid and a plane

The idea of extending the definition range of integers for x and y in (2.2) to reals leads to additional solutions: e.g.

$$1,\overline{1} = \frac{1,\overline{1}}{1}, \ (q = 10),$$

is a formal solution of Deschauer's problem (2.2), and so is the Golden Mean φ, as it solves

$$x^2 + q^{-1}xy - y = 0 \text{ for } q = -1, \ y = 1$$

$$\underbrace{1,618...}_{x} - \underbrace{1}_{y/q} = \frac{1}{x} = 0,618... \tag{2.3}$$

By the way, the two numbers 2 and 5 occur also in $1,618... = \frac{1+\sqrt{5}}{2}$!

For the Golden Mean we have the well-known representation as continued periodic fraction. By replacing the essential number "1" in the continuous fraction of the Golden Mean representation by "2", ...,"n" one receives Vera W. de Spinadel's Metallic Means [9] as generalizations of the Golden Mean. These Metallic Means are therefore solutions of quadratic equations. Any general quadratic equation

$$x^2 + rx + s = 0, \ \ (r, s \in \square), \tag{2.4}$$

can be seen as Deschauer equation (2.2), if we put $r := q^{-1}y$, $s := y$, if we admit reals instead of natural numbers. Solutions of (2.4) can be represented as periodic continuous fractions and we could use the pair of solutions of (2.4) as new coefficients of quadratic equations on one hand and on the other use the limit value of a continued fraction as the new essential (real) number in a periodic continuous fraction, too.

The first iteration process leads to an interesting "fractal set" of (in general complex) quadratic equations, e.g. the set of special equations

$$x^2 + rx - 1 = 0 \Rightarrow \{x_1,\ x_2\} \mapsto \{x^2 + x_1 x - 1 = 0,\ x^2 + x_2 x - 1 - 0\}, \ldots \tag{2.5}$$

and one could study the attractors and limit figures of this set.

The second process delivers a sequence of mean values, which can be constructed via projections of points of a hyperbola behaves similar to the construction of the so-called logistic parabola, see 2.6 and Fig. 2.

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} 1 = \varphi_1 = 0{,}618\ldots\ ; \ldots \frac{1}{\varphi_n + \frac{1}{\varphi_n + \frac{1}{\varphi_n + \ldots}}} 1 = \varphi_{n+1}, \ldots \tag{2.6}$$

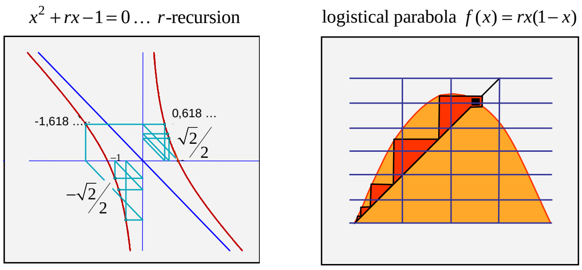

Figure 2. “logistic hyperbola” (left) visualising the r-reconstruction, compared with the logistic parabola construction (right)

3. The Cubic Mean of Hans van der Laan

The Dutch Benedictine monk, philosopher and architect Dom Hans van der Laan proposed a ratio $1 : \psi$ similar to the Golden Ratio, but it should incorporate and represent three-dimensionality. Therefore he chose ψ as the real solution of the cubic equation

$$x^3 - x - 1 = 0 \tag{3.1}$$

He coined the name “plastic number” (“design number”) for that value ψ and it turns out that this number solves an additional algebraic equation, namely

$$x^5 - x^4 - 1 = 0.$$

So the question arises to seek all such pairs of equations and call their (real) common solution a “morphic number” (see [1]).

$$\xi \text{ Morphic Number } \Leftrightarrow \xi + 1 = \xi^n \wedge \xi - 1 = \xi^{-m},\ n, m \in \square,\ n \neq 1 \tag{3.2}$$

Jan Aarts e.a. [1] found out that Laan's number ψ and the Golden Number φ (the latter in a trivial manner) are the only Morphic Numbers. Richard Padovan [7] gave a construction of ψ by a number sequence similar to the Fibonacci number ratio for the construction of φ:

$$\left.\begin{array}{l} b_0 = b_1 = b_2 = 1 \\ b_n = b_{n-3} + b_{n-2},\ n \geq 3 \\ 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots \end{array}\right\} \rightarrow \lim_{n\to\infty} \frac{b_n}{b_{n-1}} = \psi. \tag{3.3}$$